Chapitre 3

Analyse

1. Soient

$$f(x) = \begin{cases} \dfrac{x-1}{x+2} & \text{si } x \leq 4 \\[2ex] \dfrac{x-3}{2} & \text{si } x > 4 \end{cases}$$

$$g(x) = \begin{cases} \dfrac{2x-1}{x-1} & \text{si } x \leq 0 \\[2ex] \dfrac{-1}{x+2} & \text{si } x > 0 \end{cases}$$

(a) Construire les graphes de $f(x)$ et $g(x)$

(b) Construire les graphes de $(f+g)(x)$, $(f.g)(x)$ et $\left(\dfrac{f}{g}\right)(x)$ et justifier le comportement asymptotique des fonctions obtenues

(a) cf pages suivantes

(b) cf pages suivantes

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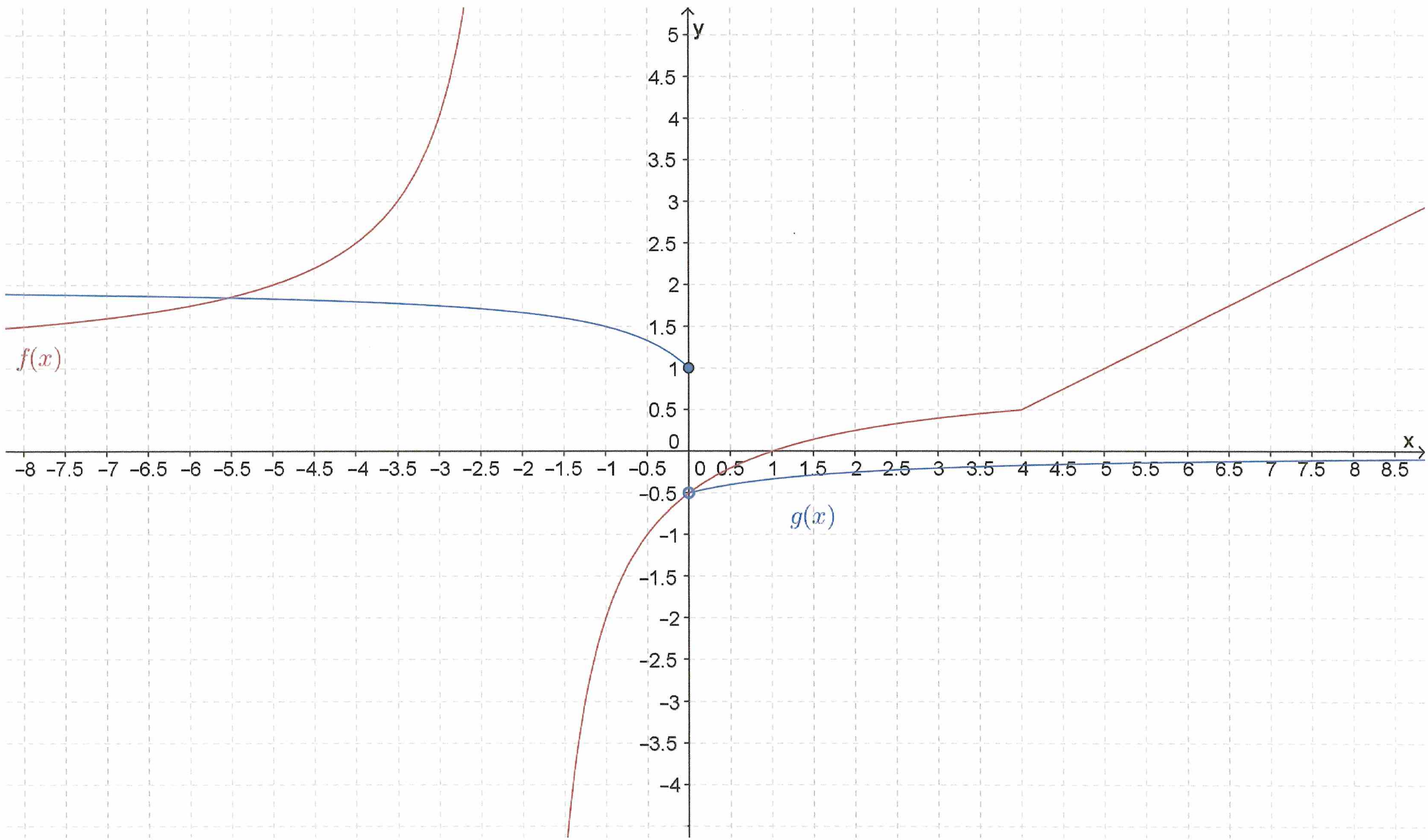
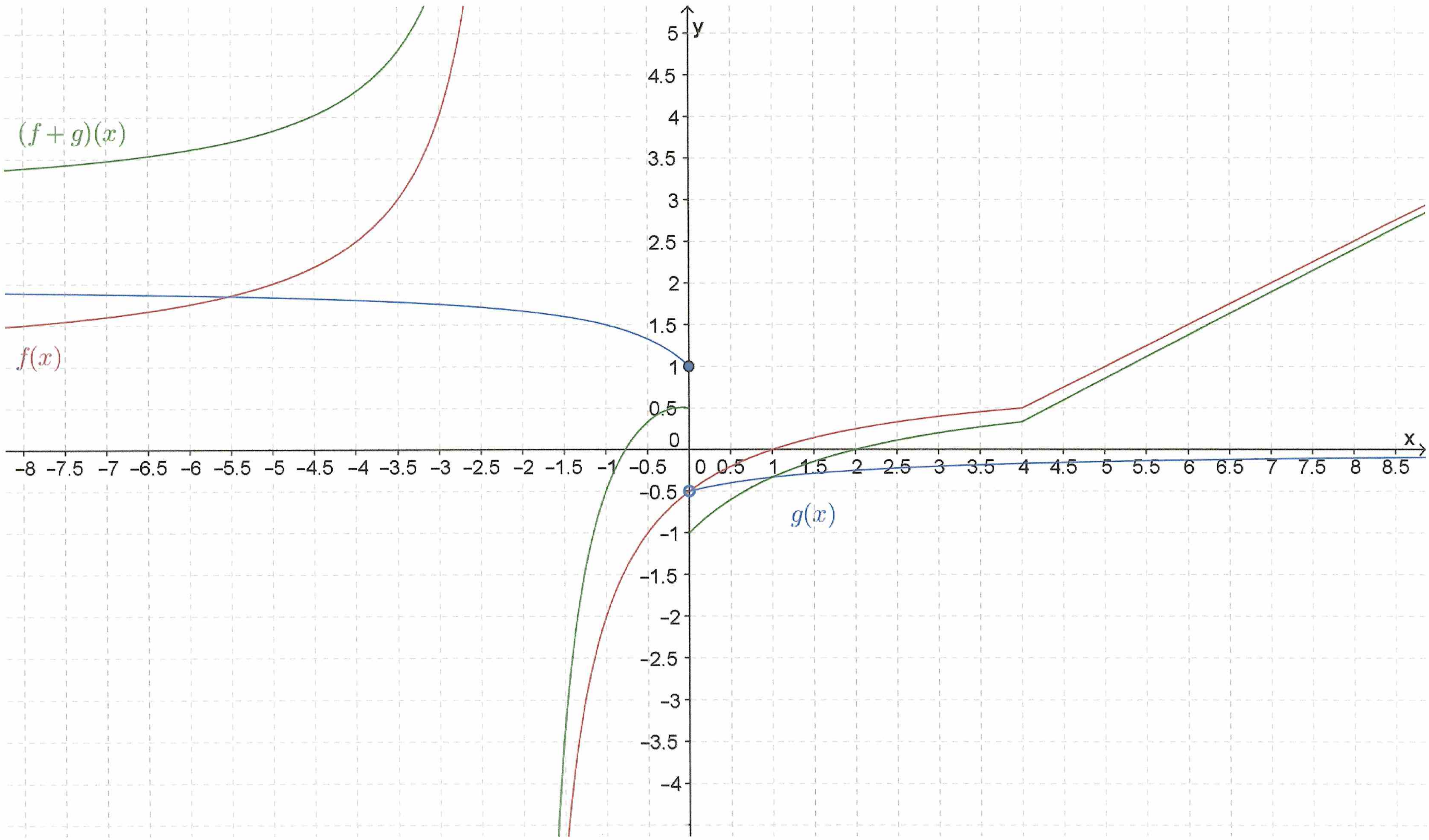
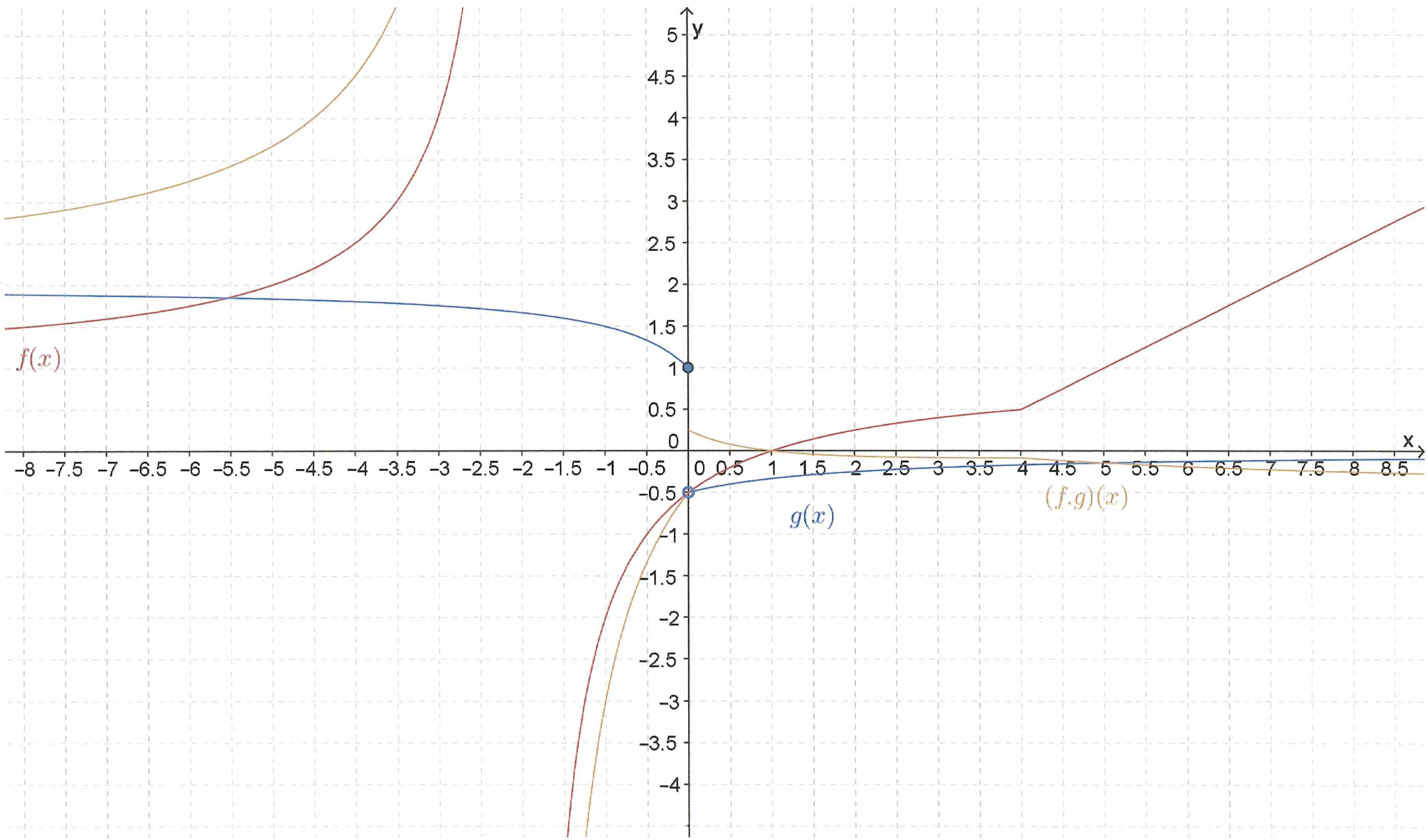

y
x
f(x)
g(x)


$(f+g)(x)$
$f(x)$
$g(x)$
x
y



f(x)
g(x)
(f.g)(x)
x
y

y
x
f(x)
(f/g)(x)
g(x)

2. Déterminer le domaine de $f(x) = \dfrac{\sqrt{-x^2+x+5}}{|x^2-2x-3|-|1-2x|}$

CE : (1) $-x^2+x+5 \geq 0$

(2) $|x^2-2x-3| - |1-2x| \neq 0$

(1) $\Delta = 1+20 = 21 \longrightarrow x_{1,2} = \dfrac{-1 \pm \sqrt{21}}{-2} = \dfrac{1 \pm \sqrt{21}}{2}$

x		$\frac{1-\sqrt{21}}{2}$		$\frac{1+\sqrt{21}}{2}$	
CE(1)	$-$	0	$+$	0	$-$

(2) $|x^2-2x-3| = \begin{cases} x^2-2x-3 & \text{si } x^2-2x-3 \geq 0 \\ -x^2+2x+3 & \text{si } x^2-2x-3 < 0 \end{cases}$ (*)

(*)

x		-1		3	
x^2-2x-3	$+$	0	$-$	0	$+$

$|1-2x| = \begin{cases} 1-2x & \text{si } x \leq \frac{1}{2} \\ -1+2x & \text{si } x > \frac{1}{2} \end{cases}$

x		-1		$\frac{1}{2}$		3	
$\lvert x^2-2x-3\rvert$	x^2-2x-3		$-x^2+2x+3$		$-x^2+2x+3$		x^2-2x-3
$\lvert 1-2x\rvert$	$1-2x$		$1-2x$		$-1+2x$		$-1+2x$
	1		2		3		4

1 et 3 : $x^2-2x-3-(1-2x) \neq 0$

$\Leftrightarrow x^2-4 \neq 0 \Leftrightarrow x \neq \pm 2$

2 et 4 : $-x^2+2x+3-(1-2x) \neq 0$

$\Leftrightarrow -x^2+4x+2 \neq 0$

$\Delta = 16+8 = 24 \qquad x_{1,2} = \dfrac{-4 \pm 2\sqrt{6}}{-2} = 2 \pm \sqrt{6}$

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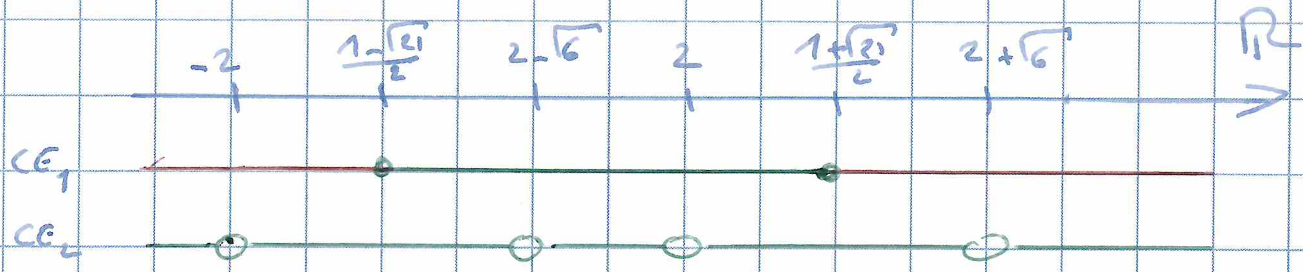

$$\text{dom } f : \left[\frac{1-\sqrt{21}}{2}, 2-\sqrt{6}\right[\cup \left]2-\sqrt{6}, 2\right[\ldots$$

$$\ldots \cup \left]2, \frac{1+\sqrt{21}}{2}\right]$$

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3. Résoudre graphiquement et vérifier algébriquement le résultat.

   (a) $(x-2)^3 < 3x - 4$

Graph : $-\infty, 1 [\cup] 1, 4 [$

Alg : $x^3 - 6x^2 + 12x - 8 - 3x + 4 < 0$

$x^3 - 6x^2 + 9x - 4 < 0$

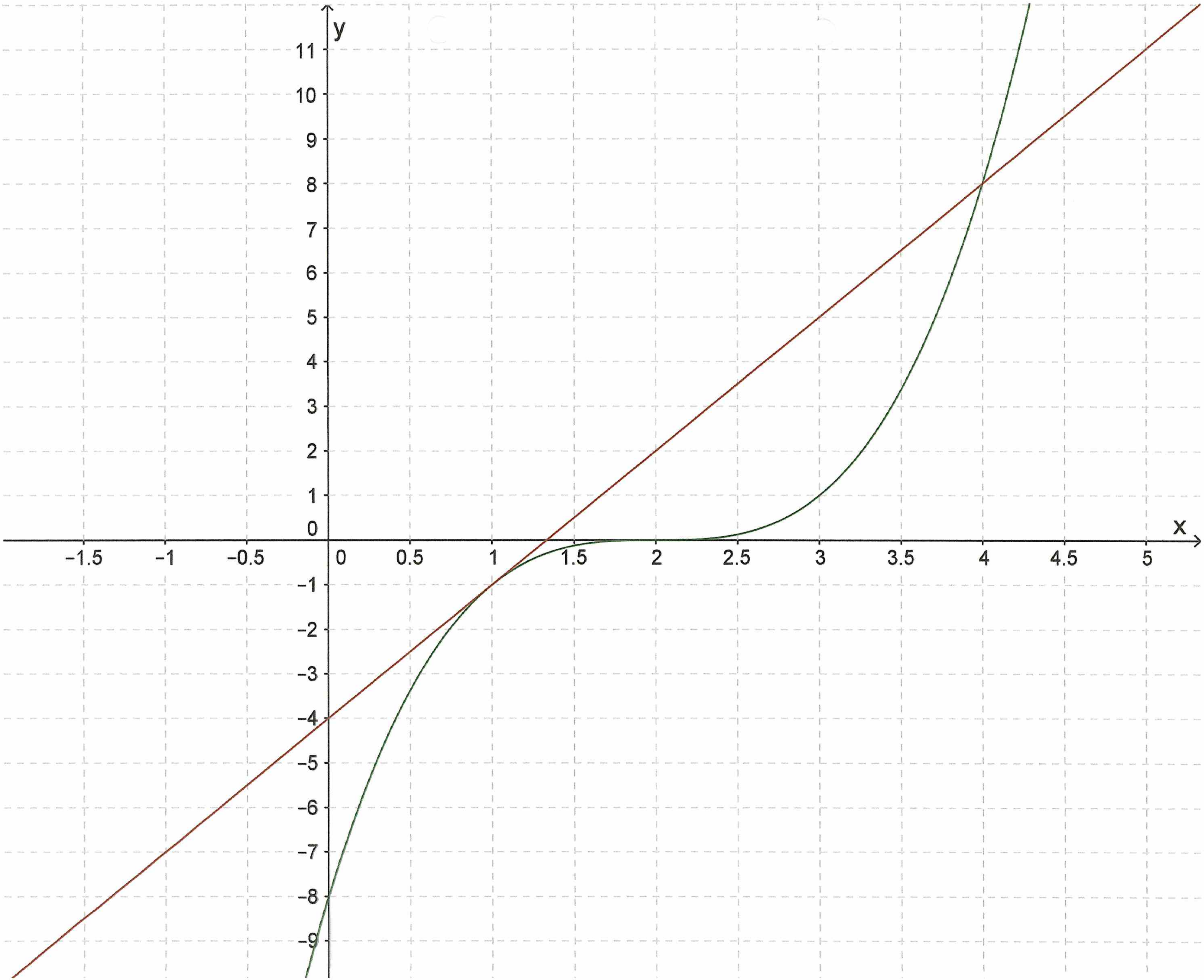
Horn : $P(1) = 0$

	1	−6	9	−4
1		1	−5	4
	1	−5	4	0

L'inéq devient : $(x-1)(x^2 - 5x + 4) < 0$

x		1		4	
$x-1$	−	0	+		+
x^2-5x+4	+	0	−	0	+
I	−	0	−	0	+

S : $-\infty, 1 [\cup] 1, 4 [$


y
x
11
10
9
8
7
6
5
4
3
2
1
0
−1
−2
−3
−4
−5
−6
−7
−8
−9
−1.5
−1
−0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

(b) $\dfrac{x}{1+|x|} \geq \dfrac{x}{2}$

Graph: $]-\infty, -1] \cup [0, 1]$

Alg: $|x| = \begin{cases} x & \text{si } x \geq 0 \\ -x & \text{si } x < 0 \end{cases}$

- $x \geq 0$ $\quad \dfrac{x}{1+x} \geq \dfrac{x}{2} \iff \dfrac{x}{1+x} - \dfrac{x}{2} \geq 0$

$$\iff \frac{2x - x - x^2}{2(1+x)} \geq 0 \iff \frac{-x^2+x}{2(1+x)} \geq 0$$

x		-1		0		1	
$-x^2+x$	$-$		$-$	0	$+$	0	$-$
$1+x$	$-$	0	$+$		$+$		$+$
Ineq	$+$	$\|$	$-$	0	$+$	0	$-$

$S_1 : [0, 1]$

- $x < 0$ $\quad \dfrac{x}{1-x} \geq \dfrac{x}{2} \iff \dfrac{x}{1-x} - \dfrac{x}{2} \geq 0$

$$\iff \frac{2x - x + x^2}{2(1-x)} \geq 0 \quad \frac{x^2+x}{2(1-x)} \geq 0$$

x		-1		0		1	
x^2+x	$+$	0	$-$	0	$+$		$+$
$1-x$	$+$		$+$		$+$	0	$-$
Ineq	$+$	0	$-$	0	$+$	$\|$	$-$

$S_2 :]-\infty, -1]$

$S = S_1 \cup S_2$

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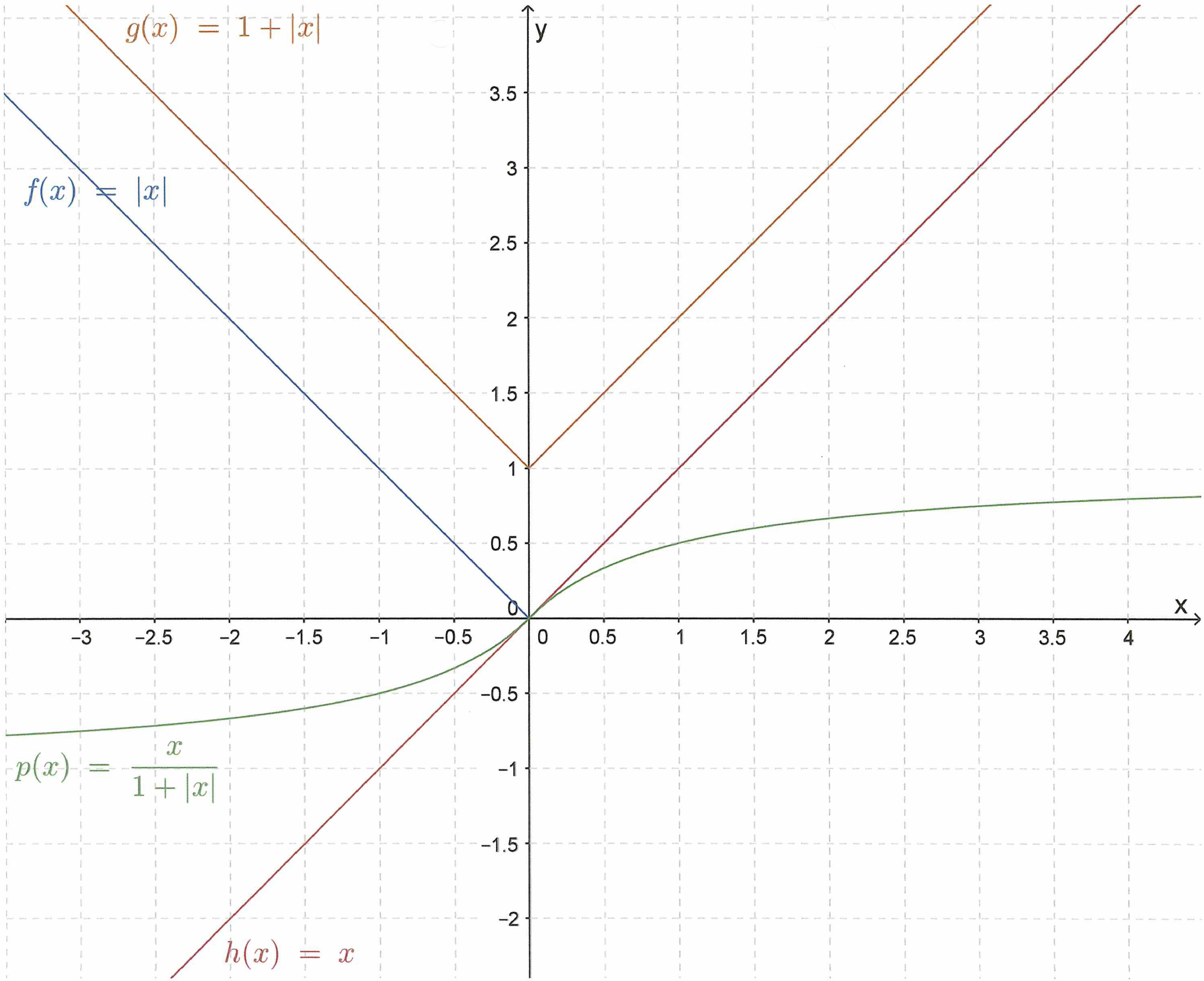
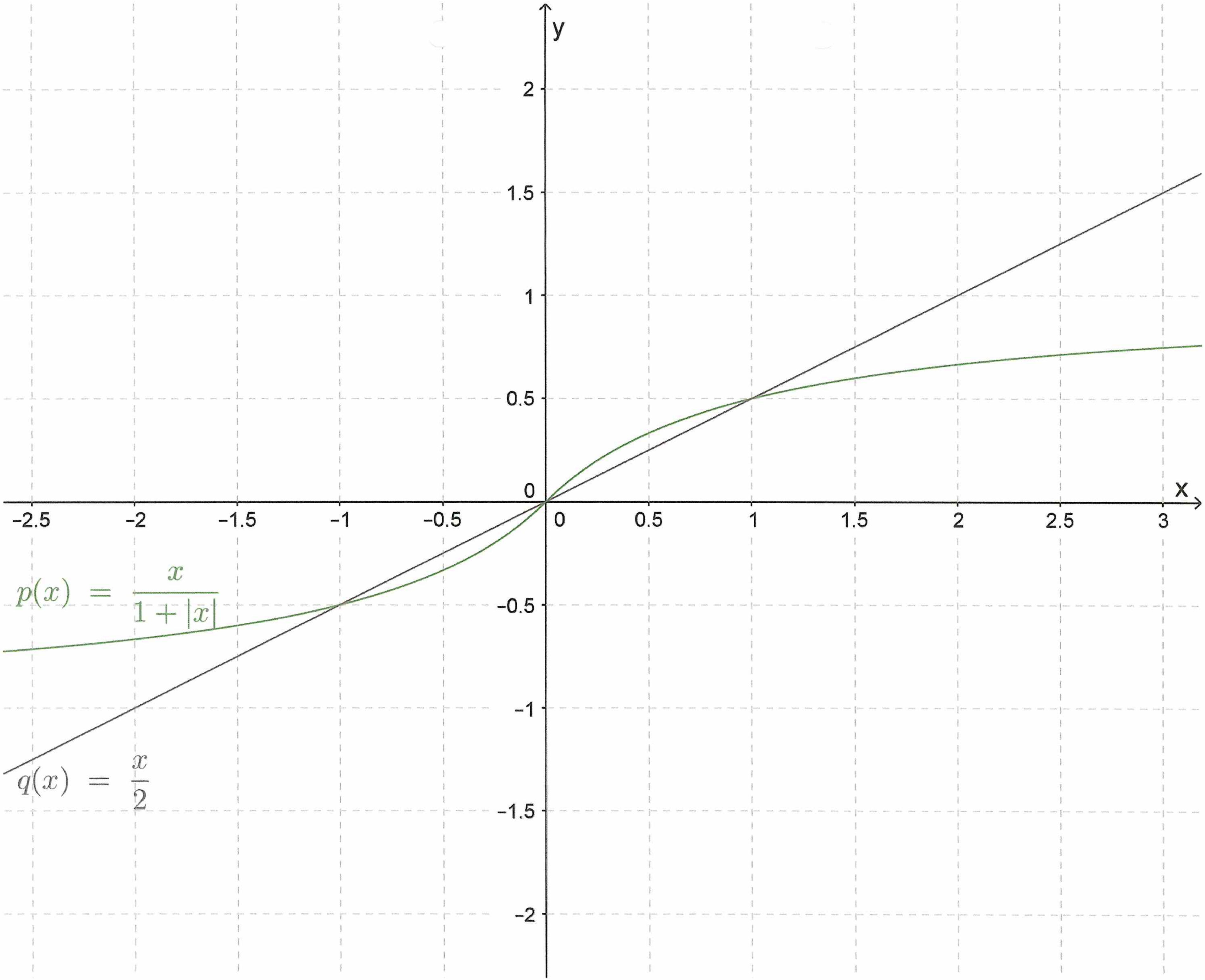


$g(x) = 1 + |x|$
$f(x) = |x|$
$p(x) = \frac{x}{1+|x|}$
$h(x) = x$
y
x



y
x
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
$p(x) = \frac{x}{1+|x|}$
$q(x) = \frac{x}{2}$

(c) $|x^2 - 1| > 1 - x$

Graph : $-\infty, -2 [\cup] 0, 1 [\cup] 1, +\infty$

Alg : $|x^2 - 1| = \begin{cases} x^2 - 1 & \text{si} \quad x \le -1 \text{ ou } x \ge 1 \\ -x^2 + 1 & \text{si} \quad -1 < x < 1 \end{cases}$

• $x \le -1$ ou $x \ge 1$

$x^2 + x - 2 > 0$ $\quad \Delta = 1 + 8 = 9$ $\quad x_{1,2} = \frac{-1 \pm 3}{2} \left\langle \begin{matrix} 1 \\ -2 \end{matrix} \right.$

x		-2	-1		1	
x^2+x-2	$+$	0		$-$	0	$+$

$S_1 : -\infty, -2 [\cup] 1, +\infty$

• $-1 < x < 1$

$-x^2 + x > 0$

x		-1	0		1	
$-x^2+x$			0	$+$	0	$-$

$S_2 :] 0, 1 [$

$S : -\infty, -2 [\cup] 0, 1 [\cup] 1, +\infty$

$= S_1 \cup S_2$

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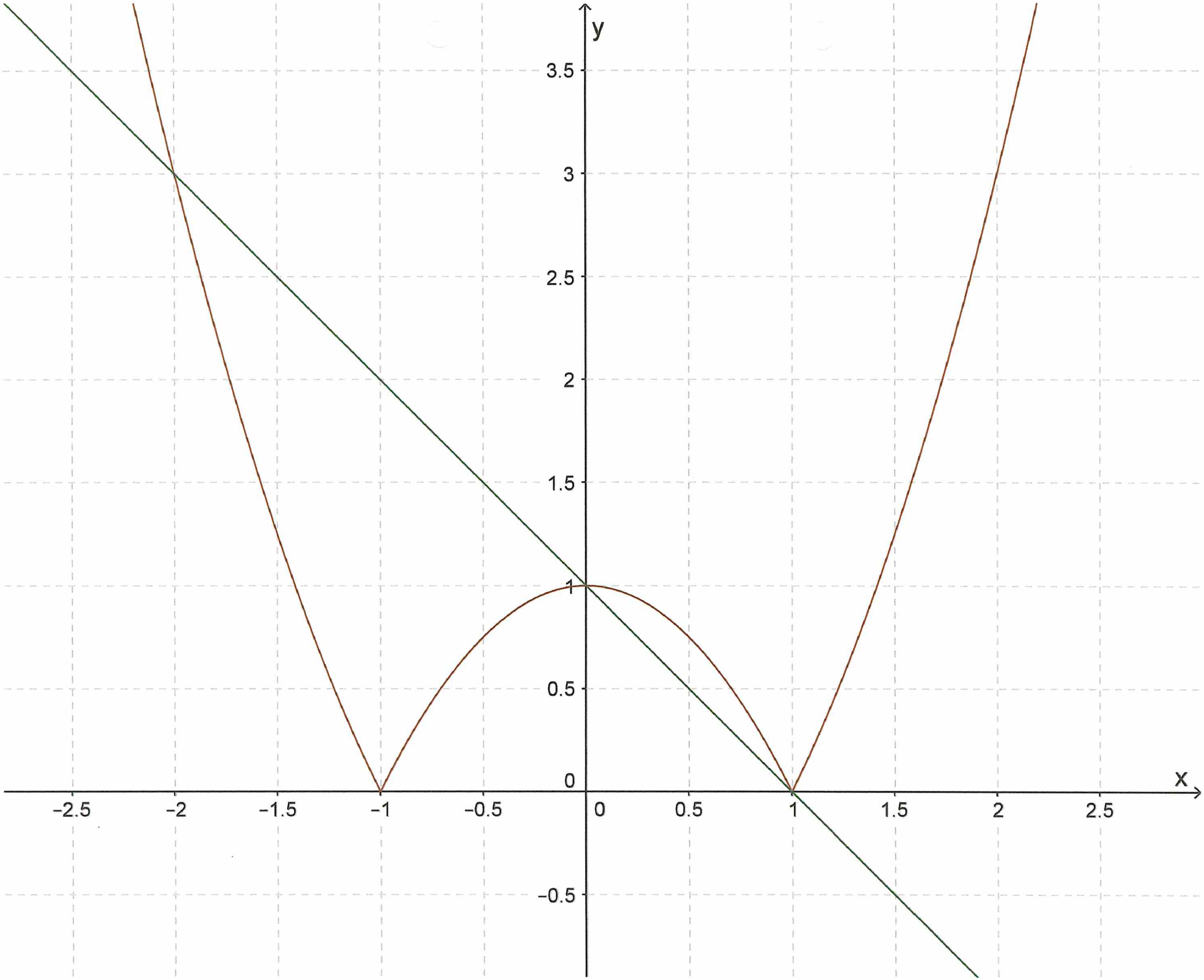
y
x
3.5
3
2.5
2
1.5
1
0.5
0
−0.5
−2.5
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
2.5

4. Résoudre $\frac{\sqrt{x(x-1)^2}}{2-x} \le \frac{1}{\sqrt{x}}$ et vérifier graphiquement le résultat.

Alg :

CE : $x>0$, $2-x\neq 0$ $\rightarrow$ $x\in]0,2[\cup]2,+\infty$

• $2-x>0 \iff x<2$

$$\frac{x(x-1)^2}{(2-x)^2}-\frac{1}{x}\le 0 \iff \frac{x^2(x^2-2x+1)-(4+x^2-4x)}{x(2-x)^2}\le 0$$

$$\iff \frac{x^4-2x^3+\cancel{x^2}-4-\cancel{x^2}+4x}{x(2-x)^2}\le 0$$

$$\iff \frac{x^4-2x^3+4x-4}{x(2-x)^2}\le 0$$

$$\iff \frac{(x^2-2)(x^2-2x+2)}{x(2-x)^2}\le 0$$

	0		$\sqrt{2}$		2	
x^2-2		$-$	0	$+$		$+$
x	0	$+$		$+$		$+$
$(2-x)^2$		$+$		$+$	0	$+$
		$-$	0	$+$	0	$+$

S_1 : $]0,\sqrt{2}]$

• $2-x<0 \iff x>0$

$$\frac{x(x-1)^2}{(2-x)^2}\ge\frac{1}{x}$$

L'inéquation est la m̄ que ci-dessus : $\frac{(x^2-2)(x^2-2x+2)}{x(2-x)^2}\ge 0$

Le tableau de signe est identique. Vu les C.E. la solution acceptable est S_2 : $]2,+\infty$

$S=S_1\cup S_2$

graph : S : $]0;1,4]\cup]2,+\infty$

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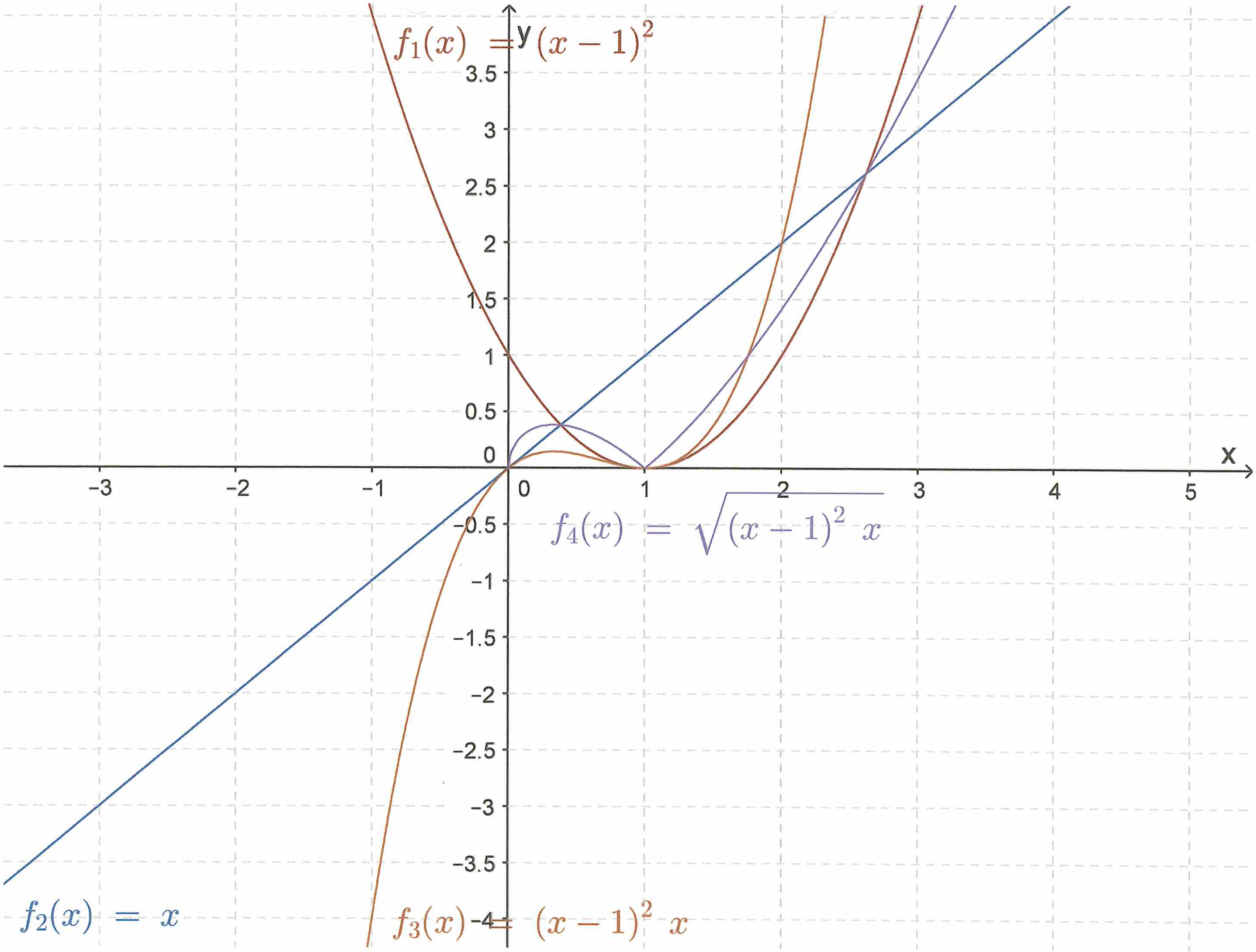
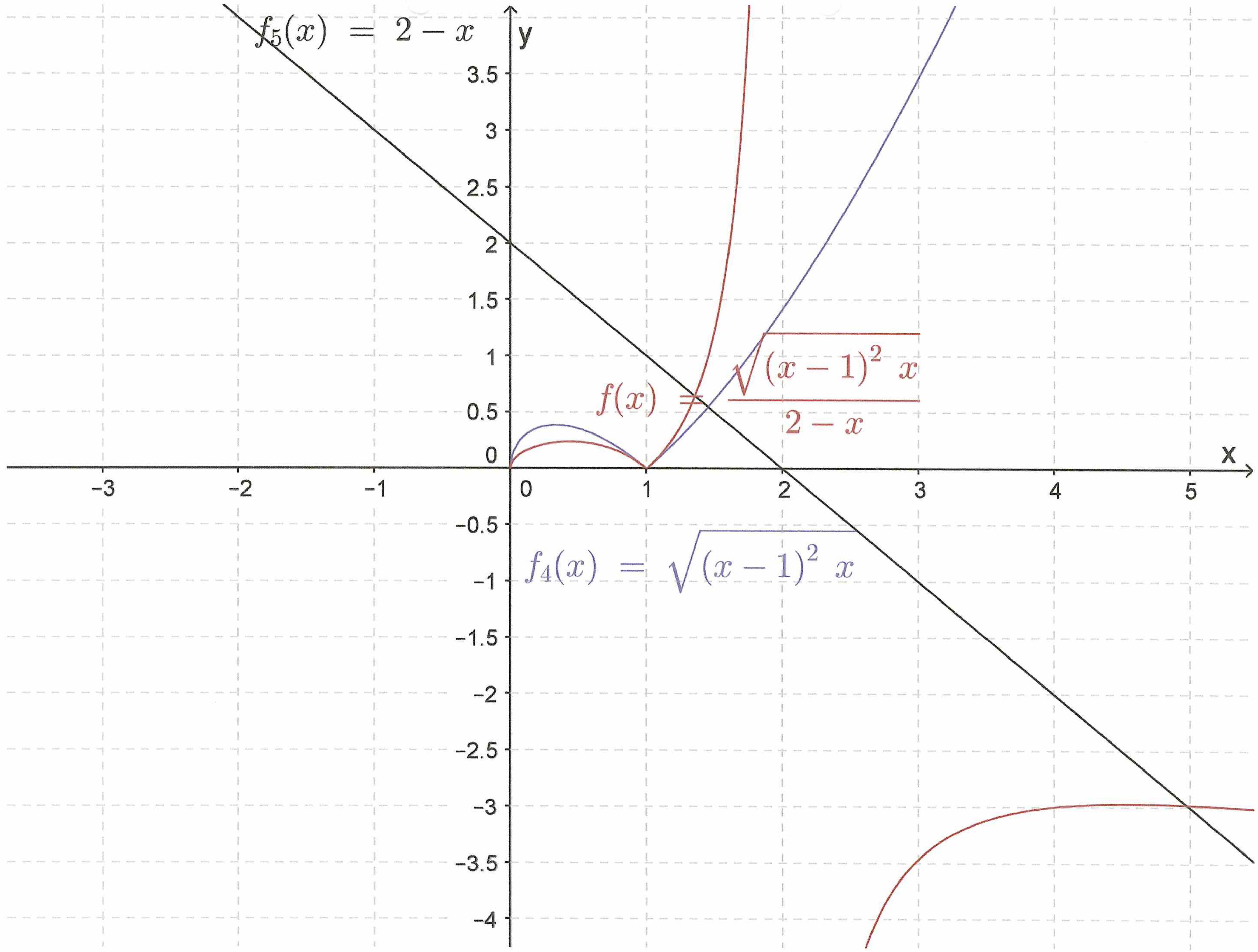
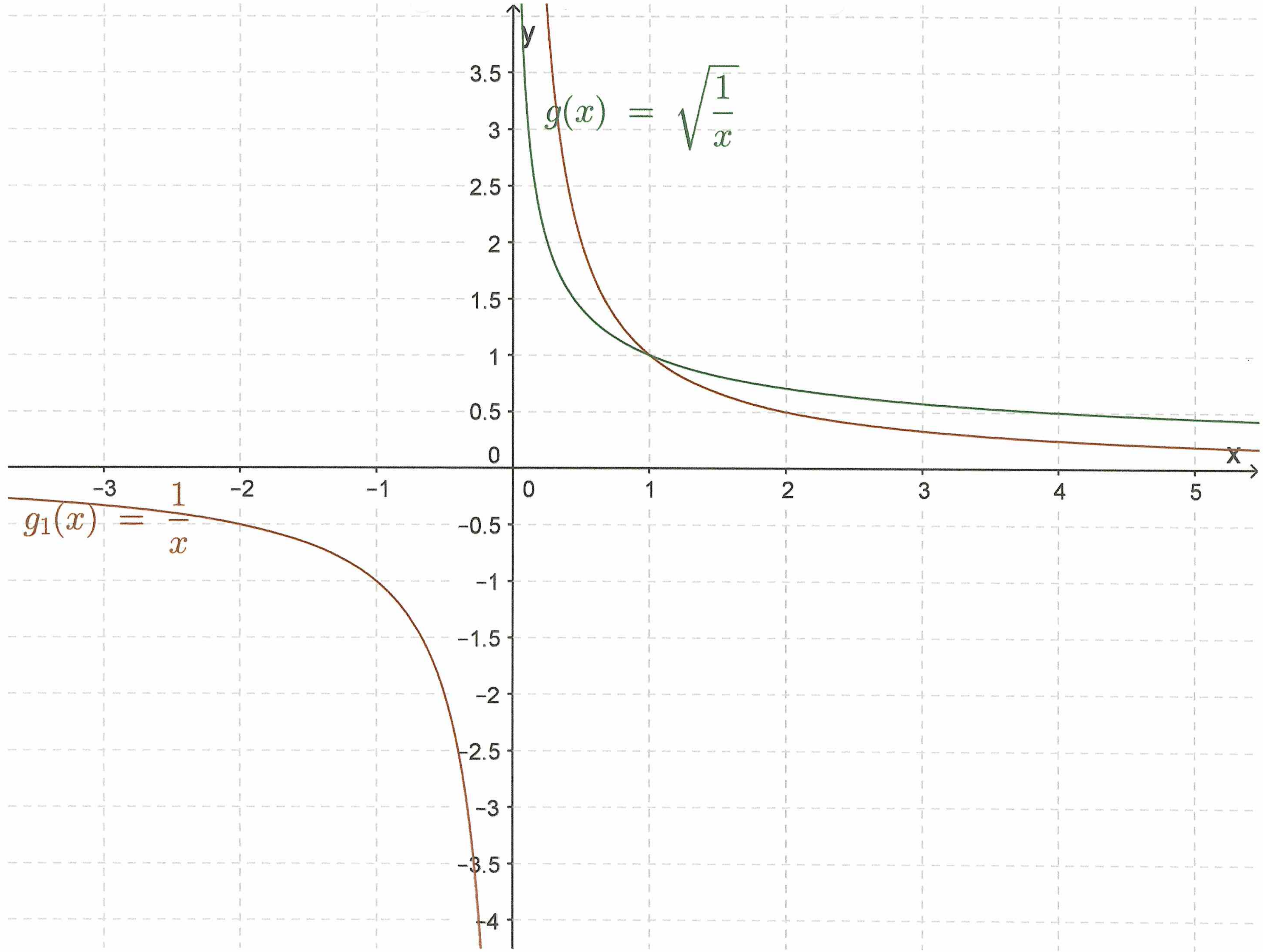


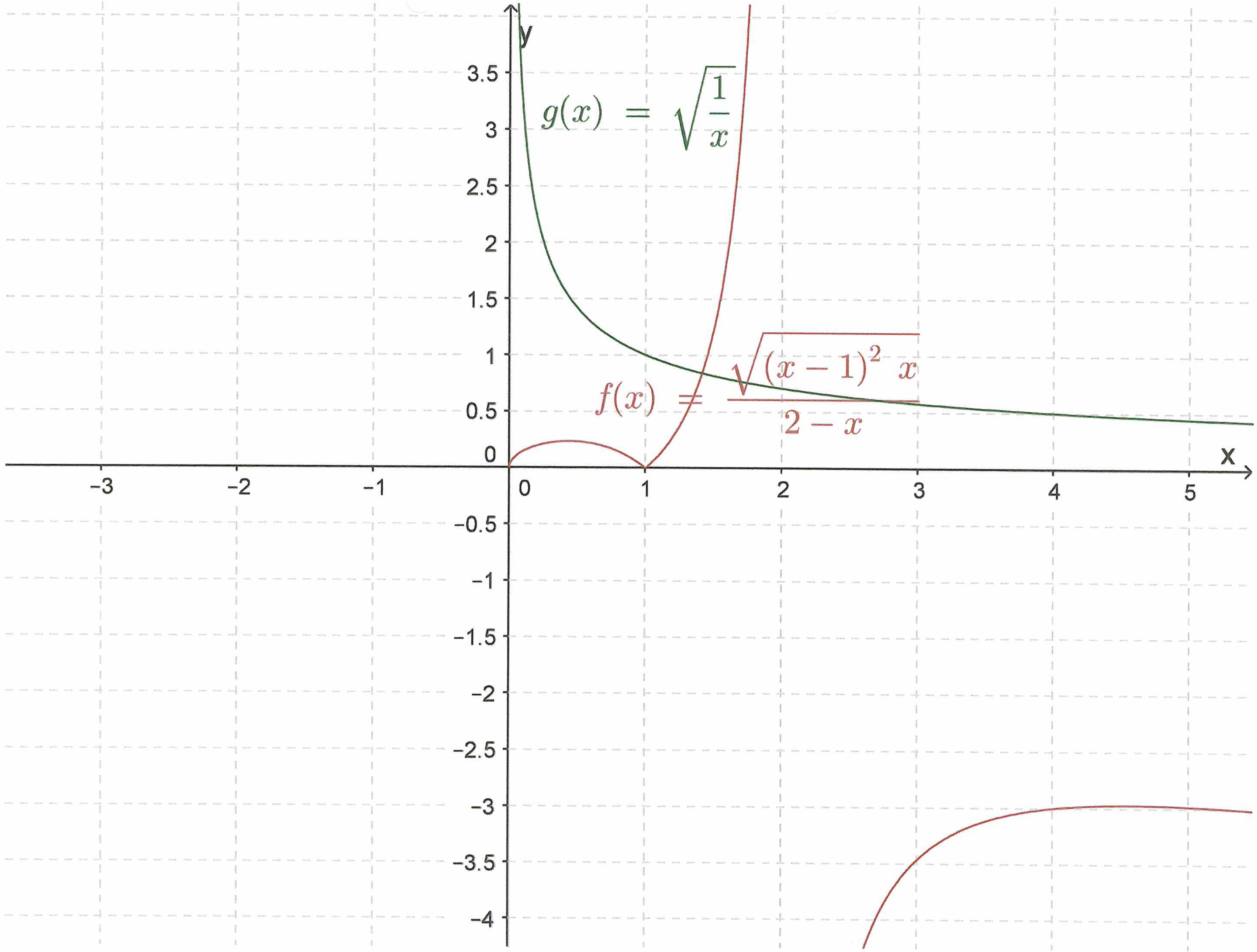
$f_1(x) = (x-1)^2$
$f_2(x) = x$
$f_3(x) = (x-1)^2\, x$
$f_4(x) = \sqrt{(x-1)^2\, x}$
x
y



$f_5(x) = 2 - x$
$f(x) = \frac{\sqrt{(x-1)^2 x}}{2-x}$
$f_4(x) = \sqrt{(x-1)^2 x}$
y
x
-3
-2
-1
0
1
2
3
4
5
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4



y
x
$g(x) = \sqrt{\frac{1}{x}}$
$g_1(x) = \frac{1}{x}$
3.5
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−3
−3.5
−4
−3
−2
−1
0
1
2
3
4
5


y
x
$g(x) = \sqrt{\frac{1}{x}}$
$f(x) = \frac{\sqrt{(x-1)^2\ x}}{2-x}$
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-3
-2
-1
0
1
2
3
4
5

5. Résoudre $x - |x^2 - 4x + 3| > 0$ et vérifier graphiquement le résultat.

Graph : $]0,7 \,;\, 4,3[$

Alg : $|x^2 - 4x + 3| = \begin{cases} x^2 - 4x + 3 & \text{si } x \leq 1 \text{ ou } x \geq 3 \\ -x^2 + 4x - 3 & \text{si } 1 < x < 3 \end{cases}$

• Si $x \leq 1$ ou $x \geq 3$:

$x - (x^2 - 4x + 3) > 0 \iff -x^2 + 5x - 3 > 0$

$\hookrightarrow \Delta = 25 - 12 = 13$

$x_{1,2} = \dfrac{-5 \pm \sqrt{13}}{-2} = \dfrac{5 \pm \sqrt{13}}{2}$

x		$\frac{5-\sqrt{13}}{2}$ $(0,69)$		1		3		$\frac{5+\sqrt{13}}{2}$ $(4,3)$	
Inég	$-$	0			$+$			0	$-$

$S_1 : \left]\dfrac{5-\sqrt{13}}{2}, 1\right] \cup \left[3, \dfrac{5+\sqrt{13}}{2}\right[$

• Si $1 < x < 3$:

$x - (-x^2 + 4x - 3) > 0 \iff x^2 - 3x + 3 > 0$

$\Delta = 9 - 12 < 0$

L'inég est toujours vraie. Vu la cond. d'existence, la solution est :

$S_2 :]1, 3[$

La solution finale est : $S = S_1 \cup S_2$

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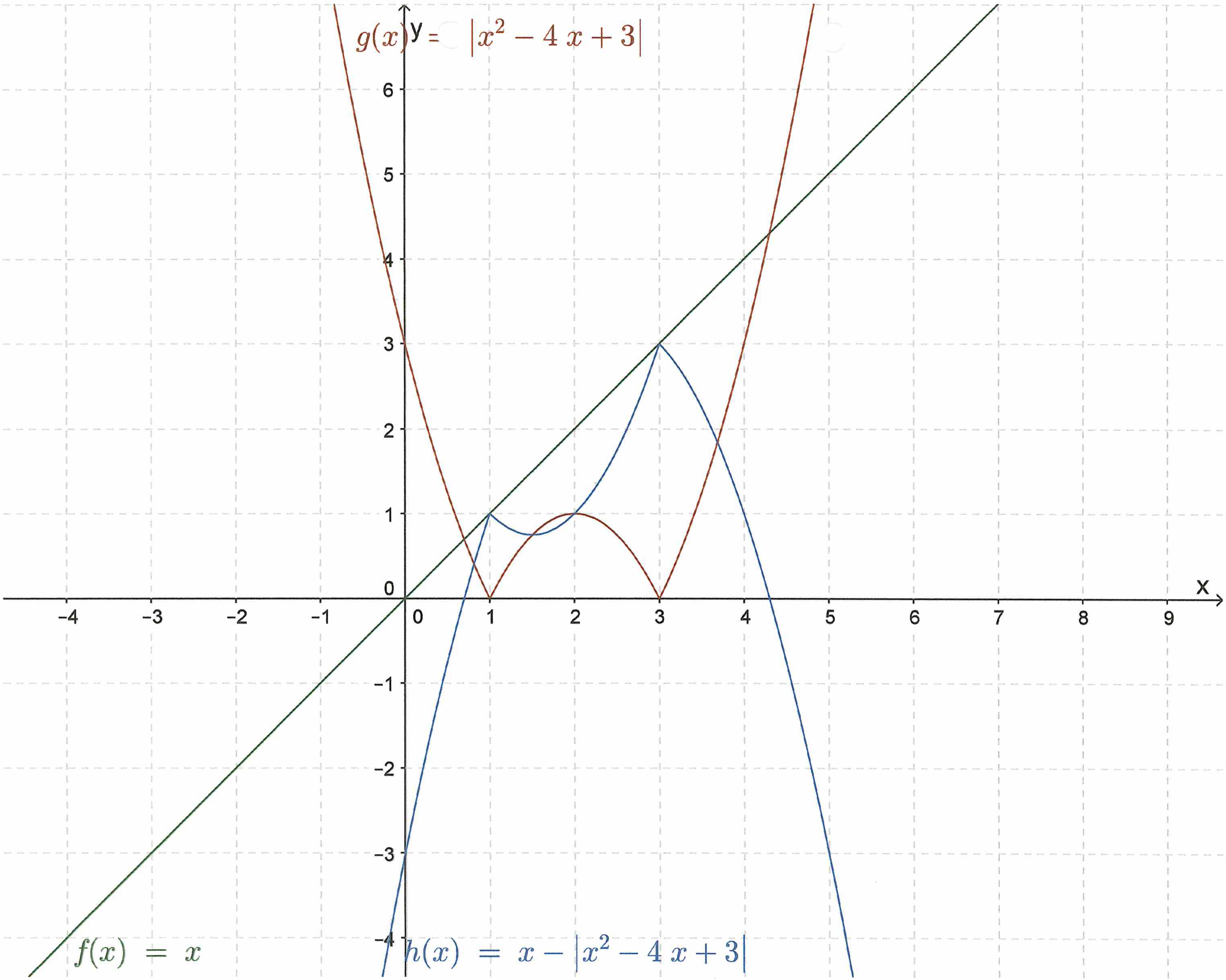

$g(x)$ y = $|x^2 - 4x + 3|$
$f(x) = x$
$h(x) = x - |x^2 - 4x + 3|$
x
y
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9

6. Soit la fonction

$$f(x) = \frac{3x+4}{x-2}$$

Tracer le graphe de $f(x)$ au départ du graphe d'une fonction de référence.
Déterminer ensuite la (les) valeur(s) de p pour laquelle (lesquelles) les droites

$$d \equiv y = -\frac{2}{5}x + p$$

sont tangentes au graphique de f.
Vérifier graphiquement les résultats et déterminer les coordonnées du point de contact à abscisse négative.

Graphe : voir annexe

Tangence $\begin{cases} y = \dfrac{3x+4}{x-2} & (1) \\ y = -\dfrac{2}{5}x + p & (2) \end{cases}$ a une solution unique

$$(1)=(2) \Leftrightarrow \frac{3x+4}{x-2} = -\frac{2}{5}x + p \Leftrightarrow 3x+4 = \left(-\frac{2}{5}x+p\right)(x-2)$$

$$\Leftrightarrow -\frac{2}{5}x^2 + \frac{4}{5}x + px - 2p - 3x - 4 = 0$$

$$\Leftrightarrow -\frac{2}{5}x^2 + \left(p - \frac{11}{5}\right)x - (4+2p) = 0$$

$$\Delta = \left(p - \frac{11}{5}\right)^2 - 4 \cdot \frac{2}{5}(4+2p) = 0 \quad (1\text{pt d'}\cap)$$

$$\Leftrightarrow p^2 + \frac{121}{25} - \frac{22}{5}p - \frac{32}{5} - \frac{16p}{5} = 0$$

$$\Leftrightarrow p^2 - \frac{38}{5}p - \frac{39}{25} = 0$$

$$\Leftrightarrow 25p^2 - 190p - 39 = 0$$

$$\hookrightarrow \Delta = 36100 + 3900 = 40000$$

$$p_{1,2} = \frac{190 \pm 200}{50} \begin{cases} \dfrac{39}{5} \\ -\dfrac{1}{5} \end{cases}$$

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Coordonnées des points d'intersection :

$$x = -\frac{b}{2a} = -\frac{\left(p - \frac{11}{5}\right)}{-\frac{4}{5}}$$

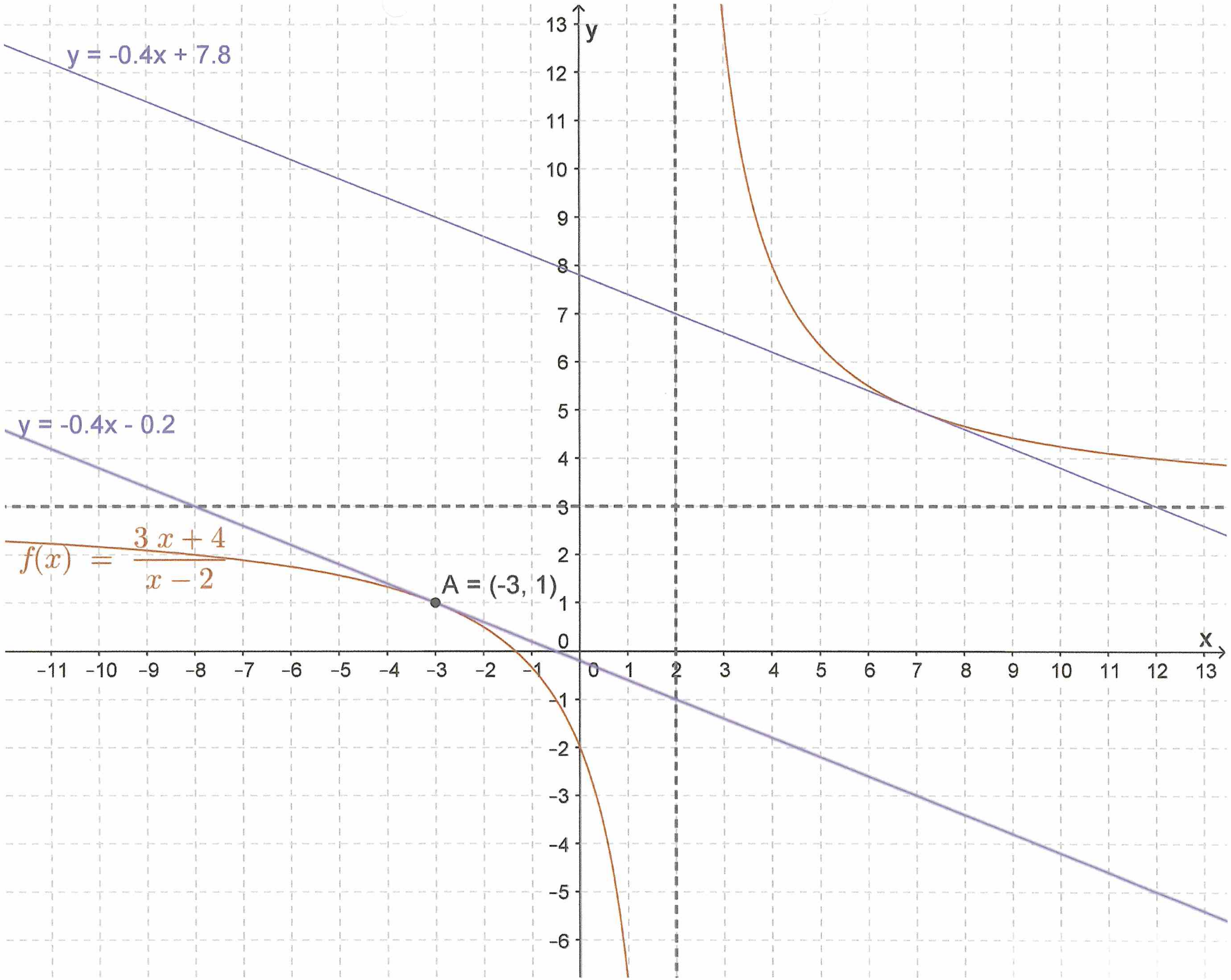
$$= \frac{5}{4}\left(p - \frac{11}{5}\right)$$

$$x < 0 \iff p = -\frac{1}{5} \rightarrow x = \frac{5}{4}\left(-\frac{1}{5} - \frac{11}{5}\right)$$

$$= -3$$

et $y = -\frac{2}{5}(-3) - \frac{1}{5} = \frac{5}{5} = 1$

$\Rightarrow$ A: $(-3, 1)$


y = -0.4x + 7.8
y = -0.4x - 0.2
$f(x) = \frac{3x+4}{x-2}$
A = (-3, 1)
x
y