Exercices complémentaires : Fonctions trigonométriques - Solutions

1. Calculer les limites suivantes :

   (a) $\lim\limits_{x\to+\infty} \dfrac{x\sin x}{x^2+3} = \dfrac{\infty \cdot ?}{\infty}$ FI

Th de l'étau : $-1 \leq \sin x \leq 1$ $\Big) \times x \ (>0)$

$-x \leq x\sin x \leq x$

$\dfrac{-x}{x^2+3} \leq \dfrac{x\sin x}{x^2+3} \leq \dfrac{x}{x^2+3}$ $\Big) \div (x^2+3)$

Or $\lim\limits_{+\infty} \dfrac{-x}{x^2+3} = \lim\limits_{+\infty} \dfrac{x}{x^2+3} = 0$ (cf 5^{B})

$\Rightarrow \lim\limits_{+\infty} \dfrac{x\sin x}{x^2+3} = 0$

   (b) $\lim\limits_{x\to 2} \dfrac{\sin(x-2)}{3x-6} = \dfrac{0}{0}$ FI

$= \lim\limits_{2} \dfrac{\sin(x-2)}{3(x-2)} = \dfrac{1}{3} \lim\limits_{2} \underbrace{\cancel{\dfrac{\sin(x-2)}{x-2}}}_{1}$

$= \dfrac{1}{3}$

(c) $\lim\limits_{x\to 0} \dfrac{x^2 \sin\left(\dfrac{1}{x}\right)}{\sin x} = \dfrac{0 \cdot ?}{0}$ FI

$= \lim\limits_{0} \dfrac{\cancel{x} \cdot x \sin \frac{1}{x}}{\cancel{\sin x}}$ car $\lim\limits_{0} \dfrac{\sin x}{x} = 1$

$= \lim\limits_{0} \left(x \cdot \sin \dfrac{1}{x} \right) = 0 \cdot ? = 0$

car $-1 \leq \underbrace{\sin \dfrac{1}{x}}_{?} \leq 1$

(d) $\lim\limits_{x\to \pi} \dfrac{\sin^2 x}{1+\cos x} = \dfrac{0}{0}$ FI

$= \lim\limits_{\pi} \left[\dfrac{\cancel{\sin^2 x}}{1+\cos x} \cdot \dfrac{1-\cos x}{1-\cos x} \right]$

$1-\cos^2 x = \cancel{\sin^2 x}$

$= \lim\limits_{\pi} (1-\cos x) = 2$

(e) $\lim_{x\to 0} \dfrac{\sin 2x}{\sqrt{1-\cos x}} = \dfrac{0}{0}$ FI

$$= \lim_{0} \left[\frac{2 \cancel{\sin} x \cos x}{\sqrt{1-\cos x}} \, \frac{\sqrt{1+\cos x}}{\sqrt{1+\cos x}} \right]$$

$$\sqrt{1-\cos^2 x} = \sqrt{\sin^2 x} = \cancel{\sin x}$$

$$= \lim_{0} \left(2 \cos x \sqrt{1+\cos x} \right)$$

$$= 2\sqrt{2}$$

(f) $\lim_{x\to 0} \dfrac{\sin^2 \frac{x}{3}}{x^2} = \dfrac{0}{0}$ FI

$$= \lim_{0} \frac{1}{9} \frac{\sin^2\left(\frac{x}{3}\right)}{\left(\frac{x}{3}\right)^2} = \frac{1}{9} \lim_{0} \frac{\sin^2 \frac{x}{3}}{\left(\frac{x}{3}\right)^2}$$

$$= \frac{1}{9}$$

(g) $\lim_{x\to 0} \frac{\sin(a+x)-\sin(a-x)}{x} = \frac{0}{0}$ FI

$$= \lim_{0} \frac{2\cos\frac{a+x+a-x}{2}\sin\frac{(a+x)-(a-x)}{2}}{x}$$

Simpson

$$= \lim_{0} \frac{2\cos a \sin x}{x}$$

$$= 2\cos a \lim_{0} \frac{\sin x}{x}$$

$$= 2\cos a$$

(h) $\lim_{x\to 0} \frac{x}{\sqrt{1-\cos x}} = \frac{0}{0}$ FI

$$= \lim_{0} \frac{\cancel{x}}{\sqrt{1-\cos x}} \frac{\sqrt{1+\cos x}}{\sqrt{1+\cos x}}$$

$$\sqrt{1-\cos^2 x} = \sqrt{\sin^2 x} = \cancel{\sin x} \qquad \lim_{0} \frac{\sin x}{x} = 1$$

$$= \lim_{0} \sqrt{1+\cos x} = \sqrt{2}$$

2. Déterminer le domaine, les zéros et les variations des fonctions suivantes :

(a) $2\sin^2 x + 2\sin x - 3$

* $\underline{\text{dom } f}$: $\mathbb{R}$
* $\underline{\text{zéros}}$: $2\sin^2 x + 2\sin x - 3 = 0$

$$\Delta = 4 + 24 = 28$$

$$\sin x_{1,2} = \frac{-2 \pm 2\sqrt{7}}{4} = \frac{-1 \pm \sqrt{7}}{2}$$

$$\sin x_1 = \frac{-1-\sqrt{7}}{2} \quad \text{imp.}$$

$$\sin x_2 = \frac{-1+\sqrt{7}}{2} \iff \begin{cases} x = 0{,}97 + 2k\pi \\ x = 2{,}18 + 2k\pi \end{cases}$$

* $f'(x) = 4\sin x \cos x + 2\cos x$
$\quad = 2\cos x\,(2\sin x + 1)$

$$f'(x) = 0 \iff \begin{cases} \cos x = 0 \iff x = \frac{\pi}{2} + k\pi \\ \sin x = -\frac{1}{2} \iff \begin{cases} x = -\frac{\pi}{6} + 2k\pi \\ x = -\frac{5\pi}{6} + 2k\pi \end{cases} \end{cases}$$

	$-\pi$		$-\frac{5\pi}{6}$		$-\frac{\pi}{2}$		$-\frac{\pi}{6}$		$\frac{\pi}{2}$		π
$\cos x$		$-$		$-$	0	$+$		$+$	0	$-$	
$2\sin x + 1$		$+$	0	$-$		$-$	0	$+$		$+$	
$f'(x)$		$-$	0	$+$	0	$-$	0	$+$	0	$-$	
$f(x)$		↘	m	↗	M	↘	m	↗	M	↘	

$\left(-\frac{5\pi}{6}, -\frac{7}{2}\right)$ $\left(-\frac{\pi}{2}, -3\right)$ $\left(-\frac{\pi}{6}, -\frac{7}{2}\right)$ $\left(\frac{\pi}{2}, 1\right)$

(b) $\dfrac{\cos x}{\sin^3 x} - 2\cot x$

* <u>dom f</u> : . $\sin x \neq 0 \iff x \neq k\pi$

. $\cot x \ \exists \iff x \neq k\pi$

dom f : $\mathbb{R} \setminus \{k\pi \mid k \in \mathbb{Z}\}$

* <u>zéros</u> : $\dfrac{\cos x}{\sin^3 x} - \dfrac{2\cos x}{\sin x} = 0$

$\iff \dfrac{\cos x - 2\cos x \sin^2 x}{\cancel{\sin^3 x}} = 0$

$\iff \cos x \left(1 - 2\sin^2 x\right) = 0$

$\iff \begin{cases} \cos x = 0 & (1) \\ 1 - 2\sin^2 x = 0 & (2) \end{cases}$

(1) $\cos x = 0 \iff x = \dfrac{\pi}{2} + k\pi$

(2) $1 - 2\sin^2 x = 0 \iff \sin^2 x = \dfrac{1}{2}$

$\iff \sin x = \pm\sqrt{\dfrac{1}{2}}$

$\iff \sin x = \pm\dfrac{\sqrt{2}}{2}$

$\sin x = \dfrac{\sqrt{2}}{2} \iff \begin{cases} x = \dfrac{\pi}{4} + 2k\pi \\ x = \dfrac{3\pi}{4} + 2k\pi \end{cases}$

$\sin x = -\dfrac{\sqrt{2}}{2} \iff \begin{cases} x = -\dfrac{\pi}{4} + 2k\pi \\ x = -\dfrac{3\pi}{4} + 2k\pi \end{cases}$

$\Rightarrow$ zéros : $\left\{ -\frac{3\pi}{4}, -\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4} \right\} + 2k\pi$ $(k \in \mathbb{Z})$

$$* \quad f'(x) = \frac{-\sin x \cdot \sin^3 x - \cos x \cdot 3 \cdot \cancel{\sin}^2 x \cos x}{\sin^{\cancel{6}\,4} x} \cdots$$

$$\cdots - 2\left(-\frac{1}{\sin^2 x}\right)$$

$$= \frac{-\sin^2 x - 3\cos^2 x + 2\sin^2 x}{\sin^4 x}$$

$$= \frac{\sin^2 x - 3(1 - \sin^2 x)}{\sin^4 x}$$

$$= \frac{4\sin^2 x - 3}{\sin^4 x}$$

$$f'(x) = 0 \Leftrightarrow 4\sin^2 x - 3 = 0 \Leftrightarrow \sin^2 x = \frac{3}{4}$$

$$\Leftrightarrow \sin x = \pm\frac{\sqrt{3}}{2}$$

$$\Leftrightarrow \begin{cases} x = \pm\frac{\pi}{3} + 2k\pi \\ x = \pm\frac{2\pi}{3} + 2k\pi \end{cases}$$

x	$-\pi$		$-\frac{2\pi}{3}$		$-\frac{\pi}{3}$		0		$\frac{\pi}{3}$		$\frac{2\pi}{3}$		π
$4\sin^2 x - 3$		$-$	0	$+$	0	$-$		$-$	0	$+$	0	$-$	
$\sin^4 x$	0	$+$		$+$		$+$	0	$+$		$+$		$+$	0
$f'(x)$	$\nexists$	$-$	0	$+$	0	$-$	$\nexists$	$-$	0	$+$	0	$-$	$\nexists$
$f(x)$	AV	$\searrow$	m_1	$\nearrow$	M_2	$\searrow$	AV	$\searrow$	m_3	$\nearrow$	M_4	$\searrow$	AV

$m_1 : \left(-\frac{2\pi}{3} , -\frac{2\sqrt{3}}{9} \right)$

$\Pi_2 : \left(-\frac{\pi}{3} , \frac{2\sqrt{3}}{9} \right)$

$m_3 : \left(\frac{\pi}{3} , -\frac{2\sqrt{3}}{9} \right)$

$\Pi_4 : \left(\frac{2\pi}{3} , \frac{2\sqrt{3}}{9} \right)$

(c) $\sin^3 x \cos x$

* **dom f** : $\mathbb{R}$
* **zéros** : $\sin^3 x = 0 \iff x = k\pi$

  $\cos x = 0 \iff x = \frac{\pi}{2} + k\pi$ $\Big\}\ x = \frac{k\pi}{2}$

* $f'(x) = 3\sin^2 x \cos x \cos x + \sin^3 x(-\sin x)$

$= 3\sin^2 x \cos^2 x - \sin^4 x$

$= \sin^2 x(3\cos^2 x - \sin^2 x)$

$= \sin^2 x(3\cos^2 x - 1 + \cos^2 x)$

$= \sin^2 x(4\cos^2 x - 1)$

$f'(x) = 0 \iff \begin{cases} \sin^2 x = 0 \iff x = k\pi \\ \cos^2 x = \frac{1}{4} \iff \cos x = \pm\frac{1}{2} \end{cases}$

$\iff \begin{cases} x = \pm\frac{\pi}{3} + 2k\pi \\ x = \pm\frac{2\pi}{3} + 2k\pi \end{cases}$

x	$-\pi$		$-\frac{2\pi}{3}$		$-\frac{\pi}{3}$		0		$\frac{\pi}{3}$		$\frac{2\pi}{3}$		π
$\sin^2 x$	0	+	+	+	+	+	0	+	+	+	+	+	0
$4\cos^2 x - 1$		+	0	−	0	+		+	0	−	0	+	
$f'(x)$	0	+	0	−	0	+	0	+	0	−	0	+	0
$f(x)$	TH	↗	M	↘	m	↗	TH	↗	M	↘	m	↗	TH
	$(-\pi,0)$		$\left(-\frac{2\pi}{3},\frac{3\sqrt{3}}{16}\right)$ (1)		(2)		$(0,0)$		(1)		$\left(\frac{2\pi}{3},\frac{-3\sqrt{3}}{16}\right)$ (2)		$(\pi,0)$

(d) $\left(3 - 4\cos^2 \frac{x}{2}\right)^2$ (+période)

* dom f : $\mathbb{R}$

* zéros : $3 - 4\cos^2 \frac{x}{2} = 0 \iff \cos^2 \frac{x}{2} = \frac{3}{4}$

$$\iff \cos \frac{x}{2} = \pm \frac{\sqrt{3}}{2} \iff \begin{cases} \frac{x}{2} = \pm \frac{\pi}{6} + 2k\pi \\ \frac{x}{2} = \pm \frac{5\pi}{6} + 2k\pi \end{cases}$$

$$\iff \begin{cases} x = \pm \frac{\pi}{3} + 4k\pi \\ x = \pm \frac{5\pi}{3} + 4k\pi \end{cases}$$

* Période : $\cos^2 \frac{x}{2} = \cos^2 \frac{x+T}{2}$

$$\iff \cos \frac{x}{2} = \pm \cos \frac{x+T}{2}$$

$$\iff \begin{cases} \cos \frac{x}{2} = \cos \frac{x+T}{2} \iff T = 4\pi \text{ (calcul "normal")} \\ \cos \frac{x}{2} = -\cos \frac{x+T}{2} \end{cases}$$

$$= \cos\left[\pi - \frac{x+T}{2}\right] \quad (2)$$

$$(2) \iff \begin{cases} \frac{x}{2} = \pi - \frac{x+T}{2} + 2k\pi \Rightarrow \text{A.R.} \text{ (T dépend de x)} \\ \frac{x}{2} = -\pi + \frac{x+T}{2} + 2k\pi \iff T = 2\pi \end{cases}$$

$\Rightarrow T = 2\pi$

$$* \; f'(x) = 2\left(3-4\cos^2\frac{x}{2}\right)\left[-4\cdot\cancel{2}\cos\frac{x}{2}\left(-\sin\frac{x}{2}\right)\cdot\frac{1}{\cancel{2}}\right]$$

$$= \overset{4\cdot 2}{\cancel{8}}\left(3-4\cos^2\frac{x}{2}\right)\cos\frac{x}{2}\sin\frac{x}{2}$$

$$= 4\left(3-4\cos^2\frac{x}{2}\right)\sin x$$

$$f'(x)=0 \iff \begin{cases} x = -\frac{\pi}{3} \text{ et } \frac{\pi}{3} \;(\text{zéros du } f(x)) \\ x = k\pi \;(\text{zéros de } \sin x) \end{cases}$$

	$-\pi$		$-\frac{\pi}{3}$		0		$\frac{\pi}{3}$		π
$\sin x$	0	$-$		$-$	0	$+$		$+$	0
$(\ldots)$		$+$	0	$-$		$-$	0	$+$	
$f'(x)$	0	$-$	0	$+$	0	$-$	0	$+$	0
$f(x)$	M	$\searrow$	m	$\nearrow$	M	$\searrow$	m	$\nearrow$	M
	$(-\pi, 9)$		$(-\frac{\pi}{3}, 0)$		$(0, 1)$		$(\frac{\pi}{3}, 0)$		$(\pi, 9)$

(e) $\dfrac{\tan x - 1}{\sec x} = (\tan x - 1) . \cos x$

* <u>dom f</u> : <u>CE</u> : $x \neq \frac{\pi}{2} + k\pi$ ($\tan x$ $\exists$)

dom f : $\mathbb{R} \setminus \{ \frac{\pi}{2} + k\pi \mid k \in \mathbb{Z} \}$

* <u>zéros</u>
$$\begin{cases} \tan x = 1 \iff x = \frac{\pi}{4} + k\pi \\ \cos x = 0 \iff x = \frac{\pi}{2} + k\pi \ (A.R) \end{cases}$$

$$* f'(x) = \frac{1}{\cos^2 x} \cos x + (\tan x - 1) . (-\sin x)$$
$$= \frac{1}{\cos x} - \left(\frac{\sin x}{\cos x} - 1 \right) \sin x$$
$$= \frac{1}{\cos x} - \frac{\sin^2 x}{\cos x} + \sin x$$
$$= \frac{\cos^2 x}{\cos x} + \sin x$$
$$= \cos x + \sin x$$

$f'(x) = 0 \iff x = -\frac{\pi}{4} + k\pi$ $\cancel{*}$

x	$-\pi$		$-\frac{\pi}{4}$		$\frac{3\pi}{4}$		π
$f'(x)$		$-$	0	$+$	0	$-$	
$f(x)$		$\searrow$	m	$\nearrow$	M	$\searrow$	
			$(-\frac{\pi}{4}, -\sqrt{2})$		$(\frac{3\pi}{4}, \sqrt{2})$		

(*) $\cos x + \sin x = 0$

$\Leftrightarrow \cos x = -\sin x$

$\Leftrightarrow \cos x = \sin(-x)$

$\Leftrightarrow \cos x = \cos\left(\frac{\pi}{2} - (-x)\right)$

$\Leftrightarrow \cos x = \cos\left(\frac{\pi}{2} + x\right)$

$$\Leftrightarrow \begin{cases} x = \frac{\pi}{2} + x + 2k\pi & \text{imp} \\ x = -\frac{\pi}{2} - x + 2k\pi \end{cases}$$

$\Leftrightarrow 2x = -\frac{\pi}{2} + 2k\pi$

$\Leftrightarrow x = -\frac{\pi}{4} + k\pi$