Formules de transformation : Solutions

1. A l'aide des formules d'addition, montrez que

   (a) $\cos\left(\frac{\pi}{2}+x\right) = -\sin x$

$$\cos\left(\frac{\pi}{2}+x\right) = \cos\frac{\pi}{2}\cos x - \sin\frac{\pi}{2}\sin x$$
$$= 0 \cdot \cos x - 1 \sin x$$
$$= -\sin x$$

   (b) $\sin\left(\frac{\pi}{4}-x\right) = \frac{\sqrt{2}}{2}(\cos x - \sin x)$

$$\sin\left(\frac{\pi}{4}-x\right) = \sin\frac{\pi}{4}\cos x - \cos\frac{\pi}{4}\sin x$$
$$= \frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x$$
$$= \frac{\sqrt{2}}{2}\left(\cos x - \sin x\right)$$

   (c) $\sin\left(\frac{\pi}{6}+x\right) + \cos\left(\frac{\pi}{6}+x\right) = \frac{\sqrt{3}+1}{2}\cos x + \frac{\sqrt{3}-1}{2}\sin x$

$$\sin\left(\frac{\pi}{6}+x\right) + \cos\left(\frac{\pi}{6}+x\right) = \sin\frac{\pi}{6}\cos x + \cos\frac{\pi}{6}\sin x \ldots$$
$$\ldots + \cos\frac{\pi}{6}\cos x - \sin\frac{\pi}{6}\sin x$$
$$= \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x + \frac{\sqrt{3}}{2}\cos x \ldots$$
$$\ldots - \frac{1}{2}\sin x$$
$$= \cos x\,\frac{\sqrt{3}+1}{2} + \sin x\,\frac{\sqrt{3}-1}{2}$$

2. Calculer, après décomposition des angles en la somme ou la différence de deux autres angles (remarquables), la valeur exacte de :

(a) $\cos\frac{5\pi}{12}$ $= \cos\left(\frac{\pi}{6}+\frac{\pi}{4}\right)$

$= \cos\frac{\pi}{6}\cos\frac{\pi}{4} - \sin\frac{\pi}{6}\sin\frac{\pi}{4}$

$= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} - \frac{1}{2}\frac{\sqrt{2}}{2}$

$= \frac{\sqrt{6}-\sqrt{2}}{4}$

(b) $\tan\frac{\pi}{12}$ $= \tan\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \frac{\tan\frac{\pi}{4}-\tan\frac{\pi}{6}}{1+\tan\frac{\pi}{4}\tan\frac{\pi}{6}}$

$= \frac{1-\frac{\sqrt{3}}{3}}{1+1\frac{\sqrt{3}}{3}} = \frac{3-\sqrt{3}}{3+\sqrt{3}}\cdot\frac{3-\sqrt{3}}{3-\sqrt{3}}$

$= \frac{9-6\sqrt{3}+3}{6} = \frac{12-6\sqrt{3}}{6} = 2-\sqrt{3}$

(c) $\sin 105°$ $= \sin(60° + 45°)$

$= \sin 60° \cos 45° + \sin 45° \cos 60°$

$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

3. Montrer que les expressions suivantes sont indépendantes de x :

(a) $\sin x + \sin\left(\frac{2\pi}{3} + x\right) + \sin\left(\frac{4\pi}{3} + x\right)$

$= \sin x + \sin\frac{2\pi}{3}\cos x + \cos\frac{2\pi}{3}\sin x + \ldots$

$\ldots \sin\frac{4\pi}{3}\cos x + \cos\frac{4\pi}{3}\sin x$

$= \cancel{\sin x} + \frac{\sqrt{3}}{2}\cancel{\cos x} - \frac{1}{2}\cancel{\sin x} - \frac{\sqrt{3}}{2}\cancel{\cos x} \ldots$

$\ldots - \frac{1}{2}\cancel{\sin x}$

$= 0$

(b) $\cos^2 x - 2\cos a \cos x \cos(a + x) + \cos^2(a + x)$

$= \cos^2 x - 2\cos a \cos x\,(\cos a \cos x - \sin a \sin x) + \ldots$

$\ldots (\cos a \cos x - \sin a \sin x)^2$

$= \cos^2 x - \cancel{2}\cos^2 a \cos^2 x + 2\cos a\,\cancel{\cos x \sin a \sin x} \ldots$

$+ \cancel{\cos^2 a}\cos^2 x - 2\,\cancel{\sin a}\sin x \cos a \cos x + \sin^2 a \sin^2 x$

$= (\cos^2 x - \cos^2 a \cos^2 x) + \sin^2 a \sin^2 x$

$= \cos^2 x\,(1 - \cos^2 a) + \sin^2 a \sin^2 x$

$= \cos^2 x \sin^2 a + \sin^2 a \sin^2 x$

$= \sin^2 a\,(\cos^2 x + \sin^2 x)$

$= \sin^2 a$

4. Calculer $\cos(a-b)$ et $\tan(a+b)$ si $\begin{cases} \sin a = \dfrac{\sqrt{2}}{2} \text{ et } a \in \left]\dfrac{\pi}{2}, \pi\right[\\ \cos b = -\dfrac{\sqrt{3}}{2} \text{ et } b \in \left]\pi, \dfrac{3\pi}{2}\right[\end{cases}$

- $\cos(a-b) = \cos a \cos b + \sin a \sin b$

$= \cos a \left(-\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{2}}{2} \sin b$

- $\cos^2 a = 1 - \sin^2 a \iff \cos^2 a = 1 - \frac{1}{2} = \frac{1}{2}$

$\iff \cos a = \pm \frac{\sqrt{2}}{2}$

$-$ Q_{II}

- $\sin^2 b = 1 - \cos^2 b \iff \sin^2 b = 1 - \frac{3}{4} = \frac{1}{4}$

$\iff \sin b = \pm \frac{1}{2}$

$-$ Q_{III}

- $\cos(a-b) = -\frac{\sqrt{2}}{2} \cdot \left(-\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right)$

$= \frac{\sqrt{6} - \sqrt{2}}{4}$

- $\tan a = -1$ et $\tan b = \frac{\sqrt{3}}{3}$

- $\tan(a+b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$

$= \dfrac{-1 + \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}} = \dfrac{-3+\sqrt{3}}{3+\sqrt{3}} \cdot \dfrac{3-\sqrt{3}}{3-\sqrt{3}}$

$= \dfrac{-9 + 3\sqrt{3} + 3\sqrt{3} - 3}{9 - 3}$

$= \dfrac{-12 + 6\sqrt{3}}{6}$

$= -2 + \sqrt{3}$

5. Montrer que

(a) $\cos a \sin(b-c) + \cos b \sin(c-a) + \cos c \sin(a-b) = 0$

$$I = \cos a(\sin b \cos c - \sin c \cos b) + \cos b(\sin c \cos a \ldots$$
$$\ldots - \sin a \cos c) + \cos c(\sin a \cos b - \sin b \cos a)$$
$$= \cos a \cancel{\sin b} \cos c - \cos a \cancel{\sin c} \cos b \ldots$$
$$+ \cos b \cancel{\sin c} \cos a - \cos b \cancel{\sin a} \sin c \ldots$$
$$+ \cos c \cancel{\sin a} \cos b - \cos c \cancel{\sin b} \cos a$$
$$= 0 = II$$

(b) $\dfrac{\cot a + \cot b}{\cot a - \cot b} = \dfrac{\sin(b+a)}{\sin(b\bar{a})}$

$$I = \frac{\dfrac{\cos a}{\sin a} + \dfrac{\cos b}{\sin b}}{\dfrac{\cos a}{\sin a} - \dfrac{\cos b}{\sin b}} = \frac{\dfrac{\cos a \sin b + \cos b \sin a}{\cancel{\sin a \sin b}}}{\dfrac{\cos a \sin b - \cos b \sin a}{\cancel{\sin a \sin b}}}$$
$$= \frac{\sin(a+b)}{\sin(b-a)} = II$$

6. Etablir la relation permettant de calculer $\tan(a+b+c)$ et déduire la relation liant $\tan a$, $\tan b$ et $\tan c$ lorsque a, b et c sont les angles d'un triangle.

$$\tan((a+b)+c) = \frac{\tan(a+b)+\tan c}{1-\tan(a+b)\tan c}$$

$$= \frac{\dfrac{\tan a+\tan b}{1-\tan a\tan b}+\tan c}{1-\dfrac{\tan a+\tan b}{1-\tan a\tan b}\cdot\tan c}$$

$$= \frac{\dfrac{\tan a+\tan b+\tan c-\tan a\tan b\tan c}{\cancel{1-\tan a\tan b}}}{\dfrac{1-\tan a\tan b-\tan a\tan c-\tan b\tan c}{\cancel{1-\tan a\tan b}}}$$

$$= \frac{\tan a+\tan b+\tan c-\tan a\tan b\tan c}{1-\tan a\tan b-\tan b\tan c-\tan a\tan c}$$

si $a+b+c=\pi \Rightarrow \tan(a+b+c)=0$

$\Leftrightarrow \tan a+\tan b+\tan c=\tan a\tan b\tan c$

7. Vérifier et donner les conditions d'existence des identités suivantes

(a) $\cos^4 a - \sin^4 a = \cos 2a$

$\Leftrightarrow \ (\cos^2 a - \sin^2 a)\underbrace{(\cos^2 a + \sin^2 a)}^{1} = \cos 2a$

$\Leftrightarrow \quad \cos 2a = \cos 2a$

(b) $\tan a + \cot a = \dfrac{2}{\sin 2a}$

$\Leftrightarrow \ \dfrac{\sin a}{\cos a} + \dfrac{\cos a}{\sin a} = \text{II}$

$\Leftrightarrow \ \dfrac{\sin^2 a + \cos^2 a}{\sin a \cos a} = \text{II}$

$\Leftrightarrow \ \dfrac{1 \cdot 2}{2 \sin a \cos a} = \text{II}$

$\Leftrightarrow \ \dfrac{2}{\sin 2a} = \dfrac{2}{\sin 2a}$

(c) $\dfrac{\cot a - 1}{\cot a + 1} = \dfrac{1 - \sin 2a}{\cos 2a}$

$$\frac{\frac{\cos a}{\sin a} - 1}{\frac{\cos a}{\sin a} + 1} = \frac{1 - 2\sin a \cos a}{\cos^2 a - \sin^2 a}$$

$$\Leftrightarrow \frac{\cos a - \sin a}{\cancel{\cos a + \sin a}} = \frac{1 - 2\sin a \cos a}{(\cos a - \sin a)\cancel{(\cos a + \sin a)}}$$

$$\Leftrightarrow (\cos a - \sin a)^2 = 1 - 2\sin a \cos a$$

$$\Leftrightarrow \cos^2 - 2\sin a \cos a + \sin^2 a = 1 - 2\sin a \cos a$$

$$\Leftrightarrow 1 - 2\sin a \cos a = 1 - 2\sin a \cos a$$

(d) $\cos 2a(1 + \tan a \tan 2a) = 1$

$$\Leftrightarrow \cos 2a + \cancel{\cos 2a}\,\frac{\sin a}{\cos a} \cdot \frac{\sin 2a}{\cancel{\cos 2a}} = 1$$

$$\Leftrightarrow \cos^2 a - \sin^2 a + \frac{\sin a}{\cancel{\cos a}} \cdot 2\sin a \cancel{\cos a} = 1$$

$$\Leftrightarrow \cos^2 a - \sin^2 a + 2\sin^2 a = 1$$

$$\Leftrightarrow \cos^2 a + \sin^2 a = 1$$

(e) $\dfrac{1+\cot^2 a}{2\cos a} = \dfrac{1}{\sin 2a \sin a}$

$$\Leftrightarrow \frac{\frac{1}{\sin^2 a}}{2\cos a} = \frac{1}{2\sin a \cos a \sin a}$$

$$\Leftrightarrow \frac{1}{2\sin^2 a \cos a} = \frac{1}{2\sin^2 a \cos a}$$

(f) $\tan\left(\dfrac{\pi}{4}+a\right) - \tan\left(\dfrac{\pi}{4}-a\right) = 2\tan 2a$

$$\Leftrightarrow \frac{\overset{1}{\tan\frac{\pi}{4}}+\tan a}{1-\tan\frac{\pi}{4}\tan a} - \frac{\tan\frac{\pi}{4}-\tan a}{1+\tan\frac{\pi}{4}\tan a} = 2\tan 2a$$

$$\Leftrightarrow \frac{(1+\tan a)^2-(1-\tan a)^2}{\cancel{1-\tan^2 a}} = \frac{4\tan a}{\cancel{1-\tan^2 a}}$$

$$\Leftrightarrow 1+2\tan a+\tan^2 a-(1-2\tan a+\tan^2 a) = 4\tan a$$

$$\Leftrightarrow 4\tan a = 4\tan a$$

(g) $\sin 2a = \dfrac{2}{\tan a + \cot a}$

$$\Leftrightarrow \quad 2 \sin a \cos a = \frac{2}{\dfrac{\sin a}{\cos a} + \dfrac{\cos a}{\sin a}}$$

$$\Leftrightarrow \quad \mathrm{I} = \frac{2}{\dfrac{\sin^2 a + \cos^2 a}{\sin a \cos a}}$$

$$\Leftrightarrow \quad 2 \sin a \cos a = 2 \sin a \cos a$$

(h) $\dfrac{\sin 2a}{1+\cos 2a} \cdot \dfrac{\cos a}{1+\cos a} = \tan \dfrac{a}{2}$

$$\Leftrightarrow \quad \frac{\cancel{2} \sin a \cancel{\cos} a}{\cancel{1} + \cancel{2} \cos^{\cancel{2}} a \cancel{-1}} \cdot \frac{\cos a}{1 + \cos a} = \tan \frac{a}{2}$$

$$\Leftrightarrow \quad \frac{\sin a}{\cancel{\cos a}} \cdot \frac{\cancel{\cos a}}{1 + \cos a} = \tan \frac{a}{2}$$

$$\Leftrightarrow \quad \frac{\cancel{2} \sin \frac{a}{2} \cancel{\cos} \frac{a}{2}}{\cancel{2} \cos^{2} \frac{a}{2}} = \tan \frac{a}{2}$$

$$\Leftrightarrow \quad \tan \frac{a}{2} = \tan \frac{a}{2}$$

8. Exprimer $\cos 3a$ en fonction de $\cos a$. Si $\tan a = 3$ et $a \in Q_I$, calculer $\cos 3a$.

$$\cos 3a = \cos(2a + a)$$
$$= \cos 2a \cos a - \sin 2a \sin a$$
$$= (2\cos^2 a - 1)\cos a - 2 \sin a \cos a \sin a$$
$$= 2\cos^3 a - \cos a - 2\cos a \sin^2 a$$
$$= 2\cos^3 a - \cos a - 2\cos a(1 - \cos^2 a)$$
$$= 2\cos^3 a - \cos a - 2\cos a + 2\cos^3 a$$
$$= 4\cos^3 a - 3\cos a$$

si $\tan a = 3 \Leftrightarrow 1 + \tan^2 a = \dfrac{1}{\cos^2 a}$

$$\Leftrightarrow 1 + 9 = \frac{1}{\cos^2 a}$$
$$\Leftrightarrow \cos^2 a = \frac{1}{10}$$
$$\Leftrightarrow \cos a = + \frac{\sqrt{10}}{10}$$

$$\cos 3a = 4 \cdot \frac{\cancel{10}\sqrt{10}}{100\cancel{0}} - 3\frac{\sqrt{10}}{10}$$
$$= \frac{(4 - 30)\sqrt{10}}{100}$$
$$= -\frac{13\sqrt{10}}{50}$$

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9. Sachant que $\cos x = \dfrac{\sqrt{2-\sqrt{2}}}{2}$ et $x \in Q_{IV}$, calculer les nombres trigonométriques de $2x$.

$$\cos 2x = 2\cos^2 x - 1$$
$$= 2 \cdot \frac{2-\sqrt{2}}{4} - 1$$
$$= \frac{2-\sqrt{2}-2}{2} = -\frac{\sqrt{2}}{2}$$

$\frac{3\pi}{2} \le x \le 2\pi \iff 3\pi \le 2x \le 4\pi \iff \pi \le 2x \le 2\pi$

or $\cos 2x < 0 \iff \pi \le 2x \le \frac{3\pi}{2}$

$$\sin 2x = -\sqrt{1-\cos^2 2x} = -\frac{\sqrt{2}}{2}$$

$$\tan 2x = \frac{\sin 2x}{\cos 2x} = 1$$

10. Factoriser les expressions

(a) $(\sin 3a + \sin 5a) + 2\sin 4a$

$= 2 \sin \frac{8a}{2} \cos \frac{-2a}{2} + 2 \sin 4a$

$= 2 \sin 4a \cos a + 2 \sin 4a$

$= 2 \sin 4a \,(1 + \cos a)$

(b) $\cos^2 2b - \cos^2 b$

$= (\cos 2b - \cos b)(\cos 2b + \cos b)$

$= -2 \sin \frac{3b}{2} \sin \frac{b}{2} \cdot 2 \cos \frac{3b}{2} \cos \frac{b}{2}$

$= - 2 \sin \frac{3b}{2} \cos \frac{3b}{2} \cdot 2 \sin \frac{b}{2} \cos \frac{b}{2}$

$= - \sin 3b \sin b$

11. Simplifier les expressions

(a) $\dfrac{\sin 4a - \sin 2a}{\cos 6a - \cos 4a}$

$$= \frac{\cancel{2} \cos 3a \cancel{\sin a}}{-\cancel{2} \sin 5a \cancel{\sin a}}$$

$$= - \frac{\cos 3a}{\sin 5a}$$

(b) $\dfrac{\sin a + \sin 3a + \sin 5a}{\cos a + \cos 3a + \cos 5a}$

$$= \frac{2 \sin 3a \cos 2a + \sin 3a}{2 \cos 3a \cos 2a + \cos 3a}$$

$$= \frac{\sin 3a \, (\cancel{2 \cos 2a + 1})}{\cos 3a \, (\cancel{2 \cos 2a + 1})}$$

$$= \tan 3a$$

12. Démontrer, en utilisant les formules de Simpson, que :

(a) $\sin a+\sin b+\sin c-\sin(a+b+c)=4\sin\frac{b+c}{2}\sin\frac{a+b}{2}\sin\frac{a+c}{2}$

$$I = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2}+2\cos\frac{c+(a+b+c)}{2}\ldots$$
$$\ldots\sin\frac{c-(a+b+c)}{2}$$

$$= 2\sin\frac{a+b}{2}\cos\frac{a-b}{2}\cancel{+}-2\cos\frac{a+b+2c}{2}\sin\frac{+(a+b)}{2}$$

$$= 2\sin\frac{a+b}{2}\left(\cos\frac{a-b}{2}-\cos\frac{a+b+2c}{2}\right)$$

$$= 2\sin\frac{a+b}{2}\cdot\left(-2\sin\frac{\frac{a+b}{2}+\frac{a+b+2c}{2}}{2}\sin\frac{\frac{a-b}{2}-\frac{a+b+2c}{2}}{2}\right)$$

$$= +4\sin\frac{a+b}{2}\sin\frac{\cancel{2}a+\cancel{2}c}{\cancel{4}2}\sin\frac{+\cancel{2}b+\cancel{2}c}{\cancel{4}2}$$

$$= II$$

(b) $\cos\left(\frac{\pi}{5}+a\right)+\cos\left(\frac{2\pi}{5}+a\right)+\cos\left(\frac{3\pi}{5}-a\right)+\cos\left(\frac{4\pi}{5}-a\right)=0$

$$I = \left(2\cos\frac{\frac{\pi}{5}+a+\frac{3\pi}{5}-a}{2}\cos\frac{\frac{\pi}{5}+a-\frac{3\pi}{5}+a}{2}\right)+\ldots$$

$$\left(2\cos\frac{\frac{2\pi}{5}+a+\frac{4\pi}{5}-a}{2}\cos\frac{\frac{2\pi}{5}+a-\frac{4\pi}{5}+a}{2}\right)$$

$$= 2\cos\frac{\cancel{4}2\pi}{\cancel{10}5}\cos\left(a-\frac{\cancel{2}\pi}{\cancel{10}5}\right)+\cos\frac{3\pi}{5}\cos\left(a-\frac{\cancel{2}\pi}{\cancel{10}5}\right)$$

$$= 2\cos\left(a-\frac{\pi}{5}\right)\underbrace{\left(\cos\frac{2\pi}{5}+\cos\frac{3\pi}{5}\right)}_{0 \text{ car angles supplémentaires}}$$

13. Exercices récapitulatifs

(a) Si $(a+b)=\frac{\pi}{4}$, vérifier que $(1+\tan a)(1+\tan b)=2$.

$\cdot\ \tan(a+b)=1 \Leftrightarrow \frac{\tan a+\tan b}{1-\tan a \tan b}=1$

$\Leftrightarrow \tan a+\tan b=1-\tan a \tan b$

$\cdot\ (1+\tan a)(1+\tan b)=2$

$\Leftrightarrow 1+\tan a+\tan b+\tan a \tan b=2$

$\Leftrightarrow \tan a+\tan b=1-\tan a\tan b$

(b) Si $\tan x=\frac{b}{a}$, vérifier que $a\cos 2x+b\sin 2x=a$.

$a(2\cos^2 x-1)+2b\sin x\cos x=a$

$\Leftrightarrow 2a\cos^2 x+2b\sin x\cos x=2a \quad \div\cos^2 x$

$\Leftrightarrow 2a+2b\tan x=2a(1+\tan^2 x)$

$\Leftrightarrow 2a+2b\cdot\frac{b}{a}=2a\left(1+\frac{b^2}{a^2}\right)$

$\Leftrightarrow 2a+\frac{2b^2}{a}=2a+\frac{2b^2}{a^2}$ OK

(c) Calculer $\sin 2x$ sachant que $\sin x - \cos x = 0.2$.

$(\sin x - \cos x)^2 = 0{,}04$

$\Leftrightarrow \sin^2 x - 2\sin x \cos x + \cos^2 x = 0{,}04$

$\Leftrightarrow 1 - \sin 2x = 0{,}04$

$\Leftrightarrow \sin 2x = 0{,}96$

(d) Si $4\sin^2 x - 2(1+\sqrt{3})\sin x + \sqrt{3} = 0$, calculer $\sin 2x$, $\cos 2x$ et $\tan 2x$.

$\Delta = 4(1+\sqrt{3})^2 - 16\sqrt{3} = 4(1+3+2\sqrt{3}) - 16\sqrt{3}$

$= 4 + 12 + 8\sqrt{3} - 16\sqrt{3} = 4 + 12 - 8\sqrt{3}$

$= 4(1+3-2\sqrt{3}) = 4(1-\sqrt{3})^2$

$\sin x_{1,2} = \dfrac{2(1+\sqrt{3}) \pm 2(1-\sqrt{3})}{8}$ $\begin{cases} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{cases}$ (*)

$\cos 2x = 1 - 2\sin^2 x$ $\begin{cases} \frac{1}{2} & (1) \\ -\frac{1}{2} & (2) \end{cases}$

(*) $\Rightarrow 0 \leq x \leq \pi \Rightarrow 0 \leq 2x \leq 2\pi$

(**)$_1$ $\Rightarrow -\frac{\pi}{2} \leq 2x \leq \frac{\pi}{2} \Rightarrow -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$ $\Bigg\}\ x \in Q_I$

(**)$_2$ $\Rightarrow \frac{\pi}{4} \leq x \leq \frac{3\pi}{4}$

(d) Si $\tan a = 4$ ~~et $a \in Q_I$~~, calculer $\sin 4a$.

$$
\begin{aligned}
\sin 4a &= 2 \sin 2a \cos 2a \\
&= 2\,(2 \sin a \cos a)\,(2\cos^2 a - 1) \\
&= 4\,(\tan a \cos a)\,(\cos a)\,(2\cos^2 a - 1) \\
&= 4 \tan a \cos^2 a \,(2\cos^2 a - 1) \\
&= 4 \tan a \,\frac{1}{1+\tan^2 a}\left(\frac{2}{1+\tan^2 a} - 1\right)
\end{aligned}
$$

Si $\tan a = 4$,

$$
\begin{aligned}
\sin 4a &= 4 \cdot 4 \,\frac{1}{17}\left(\frac{2}{17} - 1\right) \\
&= \frac{-240}{289}
\end{aligned}
$$

(e) Vérifier que $\sin 3a = 4 \sin a \sin(60° + a) \sin(60° - a)$.

$$\sin 3a = 3 \sin a - 4 \sin^3 a \quad (\text{cf exercice } 8)$$

$$
\begin{aligned}
& 4 \sin a \sin(60° + a) \sin(60° - a) \\
&= 4 \sin a \,(\overbrace{\sin 60°}^{\frac{\sqrt{3}}{2}} \cos a + \overbrace{\cos 60°}^{\frac{1}{2}} \sin a)(\sin 60° \cos a - \cos 60° \sin a) \\
&= 4 \sin a \left(\frac{3}{4}\cos^2 a - \frac{1}{4}\sin^2 a\right) \\
&= \sin a \,(3\cos^2 a - \sin^2 a) \\
&= \sin a \,[3(1 - \sin^2 a) - \sin^2 a] \\
&= \sin a \,[3 - 4\sin^2 a] \\
&= 3 \sin a - 4 \sin^3 a \qquad \text{ok}
\end{aligned}
$$

(f) ~~(g)~~ Vérifier que $\tan 3a - \tan 2a - \tan a = \tan 3a \tan 2a \tan a$.

$$\tan 3a = \tan(2a + a) = \frac{\tan 2a + \tan a}{1 - \tan 2a \tan a}$$

$$= \frac{\dfrac{2\tan a}{1-\tan^2 a} + \tan a}{1 - \dfrac{2\tan a}{1-\tan^2 a}\tan a}$$

$$= \frac{\dfrac{2\tan a + \tan a - \tan^3 a}{\cancel{1-\tan^2 a}}}{\dfrac{1-\tan^2 a - 2\tan^2 a}{\cancel{1-\tan^2 a}}}$$

$$= \frac{3\tan a - \tan^3 a}{1 - 3\tan^2 a}$$ (*) (pg suivante)

(g) ~~(h)~~ Vérifier que $(\cos a - \cos b)^2 + (\sin a - \sin b)^2 = 4\sin^2 \dfrac{a-b}{2}$.

$$I = \left(-2\sin\frac{a+b}{2}\sin\frac{a-b}{2}\right)^2 + \left(2\cos\frac{a+b}{2}\sin\frac{a-b}{2}\right)^2$$

$$= 4\sin^2\frac{a-b}{2}\left(\sin^2\frac{a+b}{2} + \cos^2\frac{a+b}{2}\right)$$

$$= 4\sin^2\frac{a-b}{2}$$

$$= II$$

(*) $\tan 3a - \tan 2a - \tan a$

$$= \frac{\tan a\,(3 - \tan^2 a)}{1 - 3\tan^2 a} - \frac{2\tan a}{1 - \tan^2 a} - \tan a$$

$$= \tan a \,\frac{(3 - \tan^2 a)(1 - \tan^2 a) - 2(1 - 3\tan^2 a) - (1 - 3\tan^2 a)(1 - \tan^2 a)}{(1 - 3\tan^2 a)(1 - \tan^2 a)}$$

$$= \tan a \,\frac{\cancel{3} - 4\cancel{\tan^2 a} + \tan^4 a - \cancel{2} + 6\tan^2 a - \cancel{1} + 4\cancel{\tan^2 a} - 3\tan^4 a}{D}$$

$$= \tan a \,\frac{-2\tan^4 a + 6\tan^2 a}{D}$$

$$= \tan a \,\underbrace{\frac{2\tan^2 a}{(1 - \tan^2 a)}}_{\tan 2a}\,\underbrace{\frac{(3 - \tan^2 a)}{(1 - 3\tan^2 a)}}_{\tan 3a}$$

$$= \tan a \;\tan 2a \;\tan 3a$$

(h) ~~(i)~~ Montrer que $\dfrac{\sin 2a + \sin 5a - \sin a}{\cos 2a + \cos 5a + \cos a} = \tan 2a$.

$$I = \frac{\sin 2a + 2\cos 3a \sin 2a}{\cos 2a + 2\cos 3a \cos 2a}$$

$$= \frac{\sin 2a\,(\cancel{1 + 2\cos 3a})}{\cos 2a\,(\cancel{1 + 2\cos 3a})}$$

$$= \tan 2a$$

$$= II$$

(i) ~~(j)~~ Montrer que $\cos 80° + \cos 40° - \cos 20° = 0$.

$$I = 2\cos 60° \cos 20° - \cos 20°$$

$$= \cos 20° \,(2\cos 60° - 1)$$

$$= \cos 20° \,(2 \cdot \tfrac{1}{2} - 1)$$

$$= 0$$

(j) ~~(k)~~ Simplifier $\cos^2(a+b)+\cos^2(a-b)-\cos 2a\cos 2b$.

$$I = (\cos a\cos b - \sin a\sin b)^2 + (\cos a\cos b + \sin a\sin b)^2 \ldots$$
$$- (\cos^2 a - \sin^2 a)(\cos^2 b - \sin^2 b)$$
$$= \cos^2 a\cos^2 b - 2\cos a\cancel{\cos b}\sin a\sin b + \sin^2 a\sin^2 b \ldots$$
$$+ \ldots \cos^2 \cancel{a} \cancel{\cos^2 b} + 2\cos a\cancel{\cos b}\sin a\sin b + \cancel{\sin^2 b}\sin^2 b \ldots$$
$$- \cos^2 a\cancel{\cos^2 b} + \cos^2 a\sin^2 b + \cos^2 b\sin^2 a - \cancel{\sin^2 a\sin^2 b}$$
$$= \cos^2 a\cos^2 b + \sin^2 a\sin^2 b + \cos^2 a\sin^2 b + \cos^2 b\sin^2 a$$
$$= \cos^2 a\,(\underbrace{\cos^2 b + \sin^2 b}_{1}) + \sin^2 a\,(\underbrace{\cos^2 b + \sin^2 b}_{1})$$
$$= \cos^2 a + \sin^2 a$$
$$= 1$$

(k) ~~(l)~~ Sachant que $\sin(a+b)=\dfrac{4}{5}$ et $a+b\in\left]\dfrac{\pi}{2},\pi\right[$, $\cos b=\dfrac{2\sqrt{5}}{5}$ et $b\in\left]\dfrac{3\pi}{2},2\pi\right[$, calculer $\sin a$.

$$\sin a = \sin[(a+b)-b]$$
$$= \sin(a+b)\cos b + \cos(a+b)\sin b$$
$$= \frac{4}{5}\,\frac{2\sqrt{5}}{5} + \cos(a+b)\sin b$$

Par la relation fondamentale : $\begin{cases}\cos(a+b) = -\dfrac{3}{5}\\[2mm] \sin b = -\dfrac{\sqrt{5}}{5}\end{cases}$

$$\Rightarrow \sin a = \frac{8\sqrt{5}}{25} + \frac{3\sqrt{5}}{25} = \frac{11\sqrt{5}}{25}$$

(l) ~~(m)~~ Simplifier $\cos\left(\frac{\pi}{6}-b\right)\cos\left(\frac{7\pi}{6}+b\right)-\sin\left(\frac{7\pi}{6}+b\right)\sin\left(\frac{\pi}{6}-b\right)$

$$= \cos\left[\left(\frac{\pi}{6}-b\right)+\left(\frac{7\pi}{6}+b\right)\right]$$

$$= \cos\frac{8\pi}{6}$$

$$= \cos\frac{4\pi}{3}$$

$$= -\frac{1}{2}$$

(m) ~~(n)~~ Démontrer que $\dfrac{\cos 6x}{\sin 4x}+\dfrac{\sin 6x}{\cos 4x}=\dfrac{1}{2\sin 2x\cos 4x}$

$$I = \frac{\cos 6x\ \cos 4x + \sin 4x\ \sin 6x}{\sin 4x\ \cos 4x}$$

$$= \frac{\cos(6x-4x)}{2\sin 2x\ \cos 2x\ \cos 4x}$$

$$= \frac{\cancel{\cos 2x}}{2\sin 2x\ \cancel{\cos 2x}\ \cos 4x}$$

$$= II$$