Chapitre 2

Analyse

1. Calculer

   (a) $\lim_{x\to 0} \dfrac{x^2}{1-\cos 2x}$ $= \dfrac{0}{0}$ FI

$= \lim_{0} \dfrac{x^2}{1-(1-2\sin^2 x)} = \lim_{0} \dfrac{x^2}{2\sin^2 x} = \dfrac{1}{2}$

$\to 1$

   (b) $\lim_{x\to 0} \dfrac{\cos x - \sqrt{\cos 2x}}{\sin^2 x}$ $= \dfrac{0}{0}$ FI

$= \lim_{0} \dfrac{(\cos x - \sqrt{\cos 2x})(\cos x + \sqrt{\cos 2x})}{\sin^2 x\,(\cos x + \sqrt{\cos 2x})}$

$= \lim_{0} \dfrac{\cos^2 x - \cos 2x}{D}$

$= \lim_{0} \dfrac{\cos^2 x - (\cos^2 x - \sin^2 x)}{D}$

$= \lim_{0} \dfrac{\cancel{\sin^2 x}}{\cancel{\sin^2 x}(\cos x + \sqrt{\cos 2x})} = \dfrac{1}{2}$

2. Ecrire l'équation de la tangente à la courbe d'équation $y = \dfrac{\cos x}{\sin 3x} + \cot x$ au point d'abscisse $\dfrac{5\pi}{6}$

$$f\left(\frac{5\pi}{6}\right) = \frac{\cos \frac{5\pi}{6}}{\sin \frac{5\pi}{2}} + \cot \frac{5\pi}{6}$$

$$= \frac{-\frac{\sqrt{3}}{2}}{1} - \sqrt{3}$$

$$= -\frac{3\sqrt{3}}{2}$$

$$f'(x) = \frac{-\sin x \sin 3x - \cos x \, 3 \cos 3x}{\sin^2 3x} - \frac{1}{\sin^2 x}$$

$$f'\left(\frac{5\pi}{6}\right) = \frac{-\frac{1}{2} \cdot 1 - \left(-\frac{\sqrt{3}}{2}\right) 3 \cdot 0}{1} - \frac{1}{\frac{1}{4}}$$

$$= -\frac{1}{2} - 4 = -\frac{9}{2}$$

$$t \equiv y + \frac{3\sqrt{3}}{2} = -\frac{9}{2}\left(x - \frac{5\pi}{6}\right)$$

$$\equiv y = -\frac{9}{2} x + \frac{15\pi}{4} - \frac{3\sqrt{3}}{2}$$

3. Etude complète de

   (a) $f(x) = \cos x + \cos 2x$

- dom f : $\mathbb{R}$
- $f(-x) = f(x) \to$ paire
- zéros : $2\cos^2 x + \cos x - 1 = 0$
- $T = 2\pi$

$\hookrightarrow \Delta = 9$, $\cos x = \frac{-1 \pm 3}{4} \begin{cases} \frac{1}{2} \\ -1 \end{cases}$

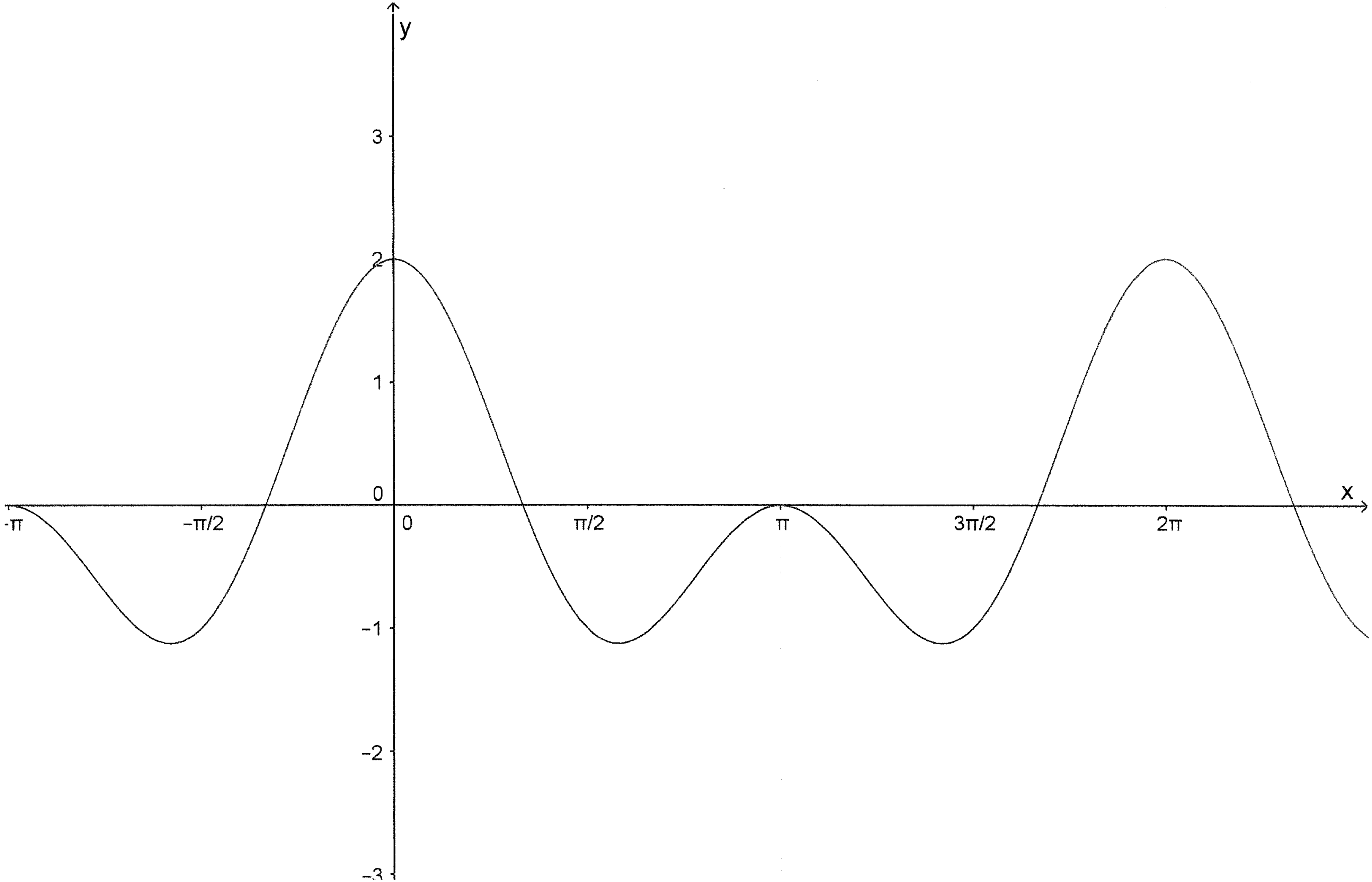
Sur $[0, \pi] \longrightarrow x = \frac{\pi}{3}$ et $x = \pi$

- pas d'asymptote
- $f'(x) = -(\sin x + 2\sin 2x)$

$= -\sin x\,(1 + 4\cos x)$

$x = 0, \pi$ $\quad$ $x = 1{,}82$ rad

x	0		$1{,}82$		π
$-\sin x$	0	$-$		$-$	0
$1+4\cos x$		$+$	0	$-$	
$f'(x)$	0	$-$	0	$+$	0
$f(x)$	M $(0,2)$	$\searrow$	m $(1{,}82\,;\,-1{,}125)$	$\nearrow$	M $(\pi, 0)$


y
3
2
1
0
-1
-2
-3
x
-π
-π/2
0
π/2
π
3π/2
2π

(b) $f(x) = \arctan(1 - x^2)$

- $\text{dom} f : \mathbb{R}$ • paire $\rightarrow \mathbb{R}^+$
- $f(x) = 0 \iff x = \pm 1$
- $\lim_{\pm\infty} f(x) = -\frac{\pi}{2} \rightarrow AH \equiv y = -\frac{\pi}{2}$
- $f'(x) = \frac{-2x}{1+(1-x^2)^2} = \frac{-2x}{x^4 - 2x^2 + 2}$ dom $\mathbb{R}^+$
- $f''(x) = \frac{-2(x^4-2x^2+2) + 2x(4x^3 - 4x)}{(x^4-2x^2+2)^2}$

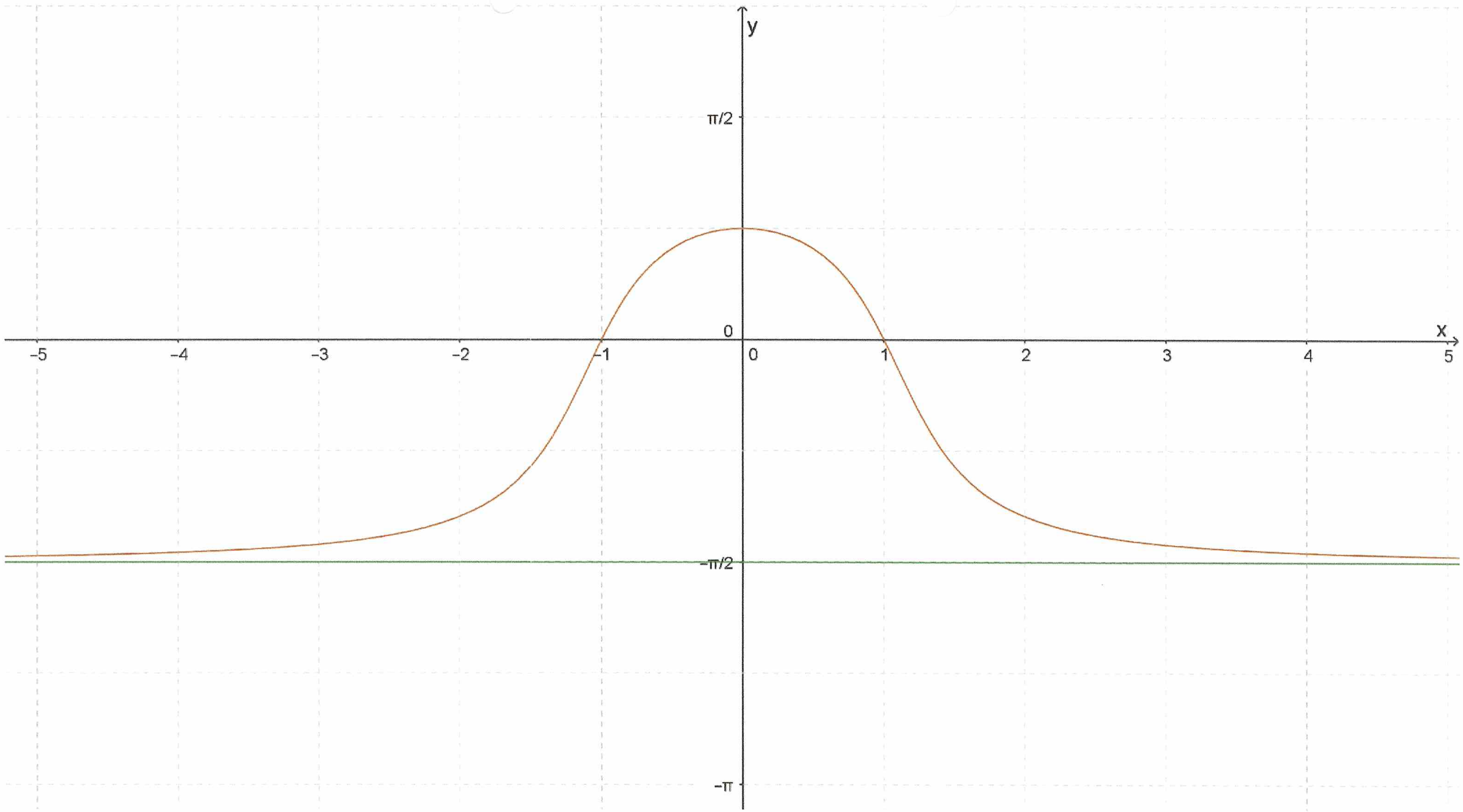
$$= \frac{-2x^4 + 4x^2 - 4 + 8x^4 - 8x^2}{D}$$

$$= \frac{6x^4 - 4x^2 - 4}{(x^4-2x^2+2)^2}$$

$$= 2\,\frac{3x^4 - 2x^2 - 2}{(x^4-2x^2+2)^2}$$

$\Delta_N = 4 + 24 = 28$ $\quad x^2_{1,2} = \frac{2 \pm 2\sqrt{7}}{6} <$ $\frac{1+\sqrt{7}}{3}$; $\frac{1-\sqrt{7}}{3}$ AR

x		0		$\sqrt{\frac{3+3\sqrt{7}}{3}}$	
$N(x)$			$-$	0	$+$
$D(x)$			$+$		$+$
$f''(x)$			$-$	0	$+$
$f(x)$				PI $(-)$	



y
π/2
0
x
-5
-4
-3
-2
-1
0
1
2
3
4
5
−π/2
−π

4. Dérivée de $x \arccos x - \sqrt{1-x^2}$

$$f'(x) = \arccos x - \cancel{\frac{x}{\sqrt{1-x^2}}} - \cancel{\frac{-2x}{2\sqrt{1-x^2}}}$$

$$= \arccos x$$

5. Calculer $\lim\limits_{x\to 0} \dfrac{\arcsin x - 2x}{\sin x}$ $= \dfrac{0}{0}$ FI

RH

$= \lim\limits_{0} \dfrac{\frac{1}{\sqrt{1-x^2}} - 2}{\cos x} = \dfrac{1-2}{1} = -1$

6. Montrer que $\arctan\frac{1}{2}+\arctan\frac{1}{5}+\arctan\frac{1}{8}=\frac{\pi}{4}$

$$\Leftrightarrow \tan\left(\arctan\frac{1}{2}+\arctan\frac{1}{5}\right)=\tan\left(\frac{\pi}{4}-\arctan\frac{1}{8}\right)$$

$$\Leftrightarrow \frac{\tan\arctan\frac{1}{2}+\tan\arctan\frac{1}{5}}{1-\tan\arctan\frac{1}{2}\,\tan\arctan\frac{1}{5}}=\frac{1-\tan\arctan\frac{1}{8}}{1+1.\tan\arctan\frac{1}{8}}$$

$$\Leftrightarrow \frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\cdot\frac{1}{5}}=\frac{1-\frac{1}{8}}{1+\frac{1}{8}}$$

$$\Leftrightarrow \frac{\frac{7}{10}}{\frac{9}{10}}=\frac{\frac{7}{8}}{\frac{9}{8}} \Leftrightarrow \frac{7}{9}=\frac{7}{9} \quad \text{ok}$$

7. Résoudre $\arctan(x-3) + \arctan x + \arctan(x+3) = \frac{5\pi}{4}$

CE :

$$\tan\left[\arctan(x-3) + \arctan x\right] = \tan\left[\frac{5\pi}{4} - \arctan(x+3)\right]$$

Par le m̂ raisonnement que la question 6 :

$$\frac{x-3+x}{1-(x-3)x} = \frac{1-(x+3)}{1+(x+3)}$$

$$\Leftrightarrow (2x-3)(x+4) = (-x-2)(-x^2+3x+1)$$

$$\Leftrightarrow 2x^2+5x-12 = x^3-x^2-7x-2$$

$$\Leftrightarrow x^3-3x^2-12x+10=0$$

$P(5)=0$

	1	−3	−12	10
5		5	10	−10
	1	2	−2	0

$x^2+2x-2=0 \rightarrow \Delta = 12$

$$x_{1,2} = \frac{-2 \pm 2\sqrt{3}}{2} = -1 \pm \sqrt{3}$$

$$S : \left\{-1-\sqrt{3}, -1+\sqrt{3}, 5\right\}$$

8. A quelle(s) condition(s) $f(x) = \dfrac{ax+b}{cx+d}$ $(c \neq 0)$ est-elle égale à sa réciproque ?

$$y = \frac{ax+b}{cx+d} \xrightarrow{\text{réciproque}} x = \frac{ay+b}{cy+d}$$

$$\Leftrightarrow cxy + dx - ay - b = 0$$

$$\Leftrightarrow y(cx-a) = b - dx$$

$$\Leftrightarrow y = \frac{-dx+b}{cx-a}$$

$\Rightarrow$ la condition est : $a = -d$

9. Asymptotes de $f(x) = x^2 \arctan \frac{1}{1+x}$

dom f : $\mathbb{R} \setminus \{-1\}$

$\lim\limits_{-1} f(x) = \pm \frac{\pi}{2} \rightarrow \cancel{AV}$

AH : $\lim\limits_{\pm\infty} f(x) = 0 \cdot \infty$ FI

$= \lim\limits_{\pm\infty} \dfrac{\arctan \frac{1}{1+x}}{\frac{1}{x^2}} = \frac{0}{0}$ FI

$= \lim\limits_{\pm\infty} \dfrac{\dfrac{\frac{-1}{(1+x)^2}}{1 + (\frac{1}{1+x})^2}}{-\frac{2}{x^3}}$

$= \lim\limits_{\pm\infty} \left\{ \left[\frac{+1}{\cancel{(1+x)^2}} \cdot \frac{\cancel{(1+x)^2}}{(1+x)^2+1} \right] \cdot \left(+ \frac{x^3}{2} \right) \right\}$

$= \frac{\infty}{\infty}$ FI

$= \lim\limits_{\pm\infty} \frac{x^3}{x^2} = \pm\infty \Rightarrow \cancel{AH}$

<u>AO</u> : $m = \lim_{\pm\infty} \dfrac{f(x)}{x}$

$= \lim_{\pm\infty} x \,\text{arctan}\, \dfrac{1}{1+x} = 0.\infty$ FI

$= \lim_{\pm\infty} \dfrac{\text{arctan}\,\frac{1}{1+x}}{\frac{1}{x}} = \dfrac{0}{0}$ FI

$\overset{RH}{\Rightarrow} \ldots$ (voir AH)

$= \lim_{\pm\infty} \dfrac{-1}{(1+x)^2+1} \cdot (-x^2) = \dfrac{\infty}{\infty}$

$= \lim_{\pm\infty} \dfrac{x^2}{x^2} = 1$

$p = \lim_{\pm\infty} \left[x^2 \,\text{arctan}\, \dfrac{1}{1+x} - x \right]$

$= \lim_{\pm\infty} \left[x^2 \left(\text{arctan}\, \dfrac{1}{1+x} - \dfrac{1}{x} \right) \right] = 0.\infty$ FI

$= \lim_{\pm\infty} \dfrac{\text{arctan}\,\frac{1}{1+x} - \frac{1}{x}}{\frac{1}{x^2}} = \dfrac{0}{0}$ FI

$\overset{RH}{=} \lim_{\pm\infty} \dfrac{\dfrac{\frac{-1}{(1+x)^2}}{1+\left(\frac{1}{1+x}\right)^2} + \dfrac{1}{x^2}}{-\dfrac{2}{x^3}}$

$= \lim_{\pm\infty} \left[\left(-\dfrac{1}{\cancel{(1+x)^2}} \cdot \dfrac{\cancel{(1+x)^2}}{(1+x)^2+1} + \dfrac{1}{x^2} \right) \left(-\dfrac{x^3}{2} \right) \right]$

$= \lim_{\pm\infty} \left[\left(\dfrac{-1}{x^2+x+2} + \dfrac{1}{x^2} \right) \left(-\dfrac{x^3}{2} \right) \right]$

$$= \lim_{\pm\infty} \left[\frac{\cancel{-x^2} + \cancel{x^2} + 2x + 2}{x^2(x^2+2x+2)} \cdot \left(\frac{-x^3}{2}\right) \right]$$

$$= \lim_{\pm\infty} \frac{-(x+1)x^3}{x^2(x^2+2x+2)} = \frac{\infty}{\infty}$$

$$= \lim_{\pm\infty} \frac{-x^4}{x^4} = -1$$

$$\Rightarrow AO \equiv y = x - 1$$