Exercices complémentaires : Forme algébrique des nombres complexes - Solutions

1. Soient $z_1 = 1+2i$, $z_2 = -3+4i$, $z_3 = i-2$ et $z_4 = 2i$.

   (a) Calculer $z_1 - 2z_2 + z_3$, $z_1.z_2.z_3$ et $\frac{z_1}{z_2}.\bar{z}_3$.

$$z_1 - 2z_2 + z_3 = 1 + 2i - 2(-3+4i) + i - 2$$
$$= -5i + 5$$

$$z_1 z_2 z_3 = (1+2i)(-3+4i)(i-2)$$
$$= (-3-8-2i)(i-2)$$
$$= (-11-2i)(i-2)$$
$$= 22 + 2 - 7i$$
$$= 24 - 7i$$

$$\frac{z_1}{z_2} \cdot \bar{z}_3 = \frac{1+2i}{-3+4i} \cdot (-i-2)$$
$$= \frac{(1+2i)(-3-4i)}{(9+16)} \cdot (-i-2)$$
$$= \frac{1}{25}(-3+8-10i)(-i-2)$$
$$= \frac{1}{\cancel{25}_{5}}(\cancel{5}^{1} - \cancel{10}^{2}\, i)(-i-2)$$
$$= \frac{1}{5}(1-2i)(-i-2)$$
$$= \frac{1}{5}(-2-2+3i)$$
$$= \frac{-4+3i}{5}$$

(b) Calculer les racines carrées de z_2 et z_4.

$\cdot\ z_2 : \quad u = x + iy \iff \begin{cases} x^2 - y^2 = -3 \\ 2xy = 4 \end{cases}$

$\iff \begin{cases} (1) \\ xy = 2 \end{cases} \iff \begin{cases} x^2 - \frac{4}{x^2} = -3 \quad (*) \\ y = \frac{2}{x} \end{cases}$

$(*) \quad x^4 + 3x^2 - 4 = 0$

$\Delta = 9 + 16 = 25$

$x^2_{1,2} = \frac{-3 \pm 5}{2} \left\langle \begin{matrix} 1 \\ -4 \to AR \end{matrix} \right.$

$x = \pm 1 \Rightarrow y = \pm 2 \Rightarrow u = \pm(1 + 2i)$

$\cdot\ z_4 : \quad u = x + iy \iff \begin{cases} x^2 - y^2 = 0 \\ 2xy = 2 \end{cases}$

$\iff \begin{cases} x^2 - \frac{1}{x^2} = 0 \quad (*) \\ y = \frac{1}{x} \end{cases}$

$(*) \quad x^4 - 1 = 0 \iff x^2 = \cancel{\pm} 1 \Rightarrow x = \pm 1$

AR

$\iff y = \pm 1 \Rightarrow u = \pm(1 + i)$

2. Résoudre

(a) $2x^2 - 3x + 1 = i(3 - x)$ $\iff 2x^2 - (3-i)x + 1 - 3i = 0$

$\Delta = (3-i)^2 - 8(1-3i)$

$= 9 - 1 - 6i - 8 + 24i$

$= 18i$

racine de $18i = \pm 3(1+i)$ (cf exos 1b)

$x_{1,2} = \dfrac{3 - i \pm (3+3i)}{4} \left\langle \begin{array}{l} \dfrac{3+i}{2} \\ -i \end{array} \right.$

$S : \left\{ -i, \dfrac{3+i}{2} \right\}$

(b) $5x^2-(9-7i)x+10=5i \iff 5x^2-(9-7i)x+10-5i=0$

$$\Delta = (9-7i)^2 - 4\cdot 5(10-5i)$$
$$= 81-49-126i-200+100i$$
$$= -168-26i$$

racines de Δ : $u = x+iy \iff \begin{cases} x^2-y^2=-168 \\ 2xy=-26 \end{cases}$

$$\iff \begin{cases} x^2 - \dfrac{169}{x^2} = -168 \quad (*) \\ y = -\dfrac{13}{x} \end{cases}$$

$$(*) \iff x^4+168x^2-169=0$$
$$\Delta = 28900$$
$$x_{1,2}^2 = \frac{-168 \pm 170}{2} \left\langle \begin{array}{l} 1 \\ -169 \quad \text{AR} \end{array} \right.$$

$$x^2 = 1 \iff x = \pm 1 \iff y = \mp 13$$

$$u = \pm(1-13i)$$

$$x_{1,2} = \frac{9-7i \pm (1-13i)}{10} \left\langle \begin{array}{l} 1-2i \\ \dfrac{4+3i}{5} \end{array} \right.$$

$$S : \left\{ 1-2i,\ \frac{4+3i}{5} \right\}$$

3. Soit le polynôme $P(z) = 2z^3 + (3-10i)z^2 - (11+9i)z - 6 - 2i$

(a) Montrer que $z = 2i$ est racine du polynôme ;

$P(2i) = 0 \Leftrightarrow 2(2i)^3 + (3-10i)(2i)^2 - (11+9i)(2i) \ldots$

$\ldots - 6 - 2i = 0$

$\Leftrightarrow -16i - 12 + 40i - 22i + 18 - 6 - 2i \overset{?}{=} 0$

OK

(b) Factoriser complètement $P(z)$

	2	$3-10i$	$-11-9i$	$-6-2i$
$2i$		$4i$	$12+6i$	$6+2i$
	2	$3-6i$	$1-3i$	0

$P(z) = (z-2i)\underbrace{[2z^2 + (3-6i)z + 1-3i]}_{P'(z)}$

<u>Fact de $P'(z)$</u> :

$\Delta = (3-6i)^2 - 8(1-3i)$

$= 9 - 36 - 36i - 8 + 24i$

$= -35 - 12i$

Racines de Δ : $u = x + iy \Leftrightarrow \begin{cases} x^2 - y^2 = -35 \\ 2xy = -12 \end{cases}$

$$\Leftrightarrow \begin{cases} x^2 - \dfrac{36}{x^2} = -35 \quad (*) \\ y = -\dfrac{6}{x} \end{cases}$$

$(*)$ $x^4 + 35x^2 - 36 = 0$

S et P $\Leftrightarrow$ $x_1^2 = -36$ et $x_2^2 = 1$

↳ AR

$x = \pm 1 \Leftrightarrow y = \mp 6 \Rightarrow u = \pm(1 - 6i)$

$$z_{1,2} = \frac{-3 + 6i \pm (1 - 6i)}{4} \left\langle \begin{matrix} -\frac{1}{2} \\ -1 + 3i \end{matrix} \right.$$

$$P(z) = \frac{1}{2}(z - 2i)\left(z + \frac{1}{2}\right)(z + 1 - 3i)$$

$$= (z - 2i)(2z + 1)(z + 1 - 3i)$$