Primitives : Solutions

1. Calculer les primitives suivantes :

(a) $\displaystyle\int x^{17}\,dx = \frac{x^{18}}{18} + C$

(b) $\displaystyle\int 7\,dx = 7x + C$

(c) $\displaystyle\int \frac{4}{x^3}\,dx = 4\int x^{-3}\,dx = 4\,\frac{x^{-2}}{-2} + C = -\frac{2}{x^2} + C$

(d) $\displaystyle\int \frac{3}{x}\,dx = 3\ln|x| + C$

(e) $\displaystyle\int (3x^2 - 4x + 7)\,dx = 3\,\frac{x^3}{3} - 4\,\frac{x^2}{2} + 7x + C$

$= x^3 - 2x^2 + 7x + C$

(f) $\displaystyle\int \left(\sqrt[3]{x^2} - \frac{2}{3x^2}\right)dx = \int \left(x^{\frac{2}{3}} - \frac{2}{3}x^{-2}\right)dx$

$\displaystyle = \frac{x^{\frac{5}{3}}}{\frac{5}{3}} - \frac{2}{3}\,\frac{x^{-1}}{-1} + C$

$\displaystyle = \frac{3}{5}\sqrt[3]{x^5} + \frac{2}{3x} + C$

(g) $\int (e^x - 3^x)\, dx = e^x - \frac{3^x}{\ln 3} + C$

(h) $\int \left(\frac{2}{\sqrt{x}} - x\sqrt{2}\right) dx = \int \left(2x^{-\frac{1}{2}} - \sqrt{2}\,x\right) dx$

$= 2\frac{x^{\frac{1}{2}}}{\frac{1}{2}} - \frac{\sqrt{2}}{2}x^2 + C$

$= 4\sqrt{x} - \frac{\sqrt{2}}{2}x^2 + C$

(i) $\int (2x^2+3)^2\, dx = \int (4x^4 + 12x^2 + 9)\, dx$

$= 4\frac{x^5}{5} + \frac{12x^3}{3} + 9x + C$

$= \frac{4}{5}x^5 + 4x^3 + 9x + C$

(j) $\int 2x(3+x^2)\, dx = \int (6x + 2x^3)\, dx$

$= 3x^2 + \frac{x^4}{2} + C$

(k) $\int (x^2+1)(x^3+2x)\, dx = \int (x^5 + 3x^3 + 2x)\, dx$

$= \frac{x^6}{6} + \frac{3}{4}x^4 + x^2 + C$

(l) $\int \left(2x^2 + \frac{2}{\sqrt{x^3}} - \frac{1}{\sqrt[3]{x^5}}\right) dx = \int \left(2x^2 + 2x^{-\frac{3}{2}} - x^{-\frac{5}{3}}\right) dx$

$= \frac{2}{3} x^3 + 2 \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} - \frac{x^{-\frac{2}{3}}}{-\frac{2}{3}} + C$

$= \frac{2}{3} x^3 - \frac{4}{\sqrt{x}} + \frac{3}{2\sqrt[3]{x^2}} + C$

(m) $\int \frac{3x^3 + 2x^2 + 1}{x^2} dx$

$= \int \left(3x + 2 + x^{-2}\right) dx$

$= \frac{3}{2} x^2 + 2x - \frac{1}{x} + C$

2. Calculer les primitives suivantes :

(a) $\int \frac{2}{(2x+1)^4} dx$ $\quad g(x) = 2x+1 \quad , g'(x) = 2$

$= \int g'(x)\, g^{-4}(x)\, dx$

$= \frac{g^{-3}(x)}{-3} + C$

$= \frac{1}{-3(2x+1)^3} + C$

(b) $\int \frac{1}{5-2x}\,dx$ $\qquad g(x) = 5-2x \qquad g'(x) = -2$

$$= -\frac{1}{2}\int \frac{-2}{5-2x}\,dx$$

$$= -\frac{1}{2}\int \frac{g'}{g}\,dx$$

$$= -\frac{1}{2}\ln|g(x)| + C$$

$$= -\frac{1}{2}\ln|5-2x| + C$$

(c) $\int \frac{1}{\sqrt{3x+5}}\,dx$ $\qquad g(x) = 3x+5 \qquad g'(x) = 3$

$$= \frac{1}{3}\int \frac{3}{\sqrt{3x+5}}\,dx$$

$$= \frac{1}{3}\int g' \cdot g^{-\frac{1}{2}}\,dx$$

$$= \frac{1}{3}\cdot\frac{g^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \frac{2}{3}\sqrt{3x+5} + C$$

(d) $\int (1-2x)^7 \, dx$

$g(x) = 1-2x \qquad g'(x) = -2$

$= -\frac{1}{2} \int -2(1-2x)^7 \, dx$

$= -\frac{1}{2} \int g' g^7 \, dx$

$= -\frac{1}{2} \frac{g^8}{8} + C$

$= -\frac{1}{16} (1-2x)^8 + C$

(e) $\int \frac{2x+1}{(x^2+x+2)^3} \, dx$

$g(x) = x^2+x+2 \qquad g'(x) = 2x+1$

$= \int \frac{g'}{g^3} \, dx$

$= \frac{g^{-2}}{-2} + C$

$= -\frac{1}{2(x^2+x+2)^2} + C$

(f) $\displaystyle\int \frac{x+1}{\sqrt{(x^2+2x-8)^3}}\,dx$

$g(x) = x^2+2x-8 \qquad g'(x) = 2x+2$

$$= \frac{1}{2}\int \frac{2(x+1)}{\sqrt{(x^2+2x-8)^3}}\,dx$$

$$= \frac{1}{2}\int g' \, g^{-3/2}\,dx$$

$$= \frac{1}{2}\,\frac{g^{-\frac{1}{2}}}{-\frac{1}{2}} + C$$

$$= -\frac{1}{\sqrt{(x^2+2x-8}} + C$$

(g) $\displaystyle\int \frac{1}{x^2\sqrt{1+\frac{1}{x}}}\,dx$

$g(x) = 1+\frac{1}{x} \qquad g'(x) = -\frac{1}{x^2}$

$$= -\int \frac{-1}{x^2\sqrt{1+\frac{1}{x}}}\,dx$$

$$= -\int g' \, g^{-\frac{1}{2}}\,dx$$

$$= -\frac{g^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= -2\sqrt{1+\frac{1}{x}} + C$$

(h) $\int \frac{x+3}{2(x^2+6x+5)^4}\,dx$ $\qquad g(x) = x^2+6x+5 \qquad g'(x) = 2x+6$

$$= \frac{1}{2}\cdot\frac{1}{2}\int \frac{2x+6}{(x^2+6x+5)^4}\,dx$$

$$= \frac{1}{4}\int g'\cdot g^{-4}\,dx$$

$$= \frac{1}{4}\,\frac{g^{-3}}{-3} + C$$

$$= -\frac{1}{12(x^2+6x+5)^3} + C$$

(i) $\int \frac{1}{2x^2+3}\,dx$ $\qquad$ (arctan)

$$= \frac{1}{3}\int \frac{1}{\frac{2x^2}{3}+1}\,dx$$

$$= \frac{1}{3}\int \frac{1}{\left(\sqrt{\frac{2}{3}}x\right)^2+1}\,dx \qquad g(x) = \sqrt{\frac{2}{3}}\,x \qquad g'(x) = \sqrt{\frac{2}{3}}$$

$$= \frac{1}{3}\sqrt{\frac{3}{2}}\int \frac{\sqrt{\frac{2}{3}}}{\left(\sqrt{\frac{2}{3}}x\right)^2+1}\,dx$$

$$= \frac{\sqrt{6}}{6}\cdot\int \frac{g'}{g^2+1}\,dx$$

$$= \frac{\sqrt{6}}{6}\arctan g + C$$

$$= \frac{\sqrt{6}}{6}\arctan \frac{\sqrt{6}}{3}x + C$$

(j) $\int \frac{1}{\sqrt{16-9x^2}}\,dx$

$$= \frac{1}{4}\int \frac{1}{\sqrt{1-\frac{9x^2}{16}}}\,dx$$

$$= \frac{1}{4}\int \frac{1}{\sqrt{1-\left(\frac{3x}{4}\right)^2}}\,dx \qquad g = \frac{3x}{4} \qquad g' = \frac{3}{4}$$

$$= \frac{1}{3}\int \frac{\frac{3}{4}}{\sqrt{1-\left(\frac{3x}{4}\right)^2}}\,dx$$

$$= \frac{1}{3}\int \frac{g'}{\sqrt{1-g^2}}\,dx$$

$$= \frac{1}{3}\arcsin g + C = \frac{1}{3}\arcsin \frac{3x}{4} + C$$

(k) $\int \frac{e^x}{e^x+1}\,dx$ $\qquad g(x) = e^x+1 \qquad g'(x) = e^x$

$$= \int \frac{g'}{g}\,dx$$

$$= \ln|g| + C$$

$$= \ln(e^x+1) + C$$

(l) $\int 3xe^{x^2}\,dx$ $\qquad g = x^2 \qquad g' = 2x$

$= 3 \cdot \frac{1}{2} \int 2x\, e^{x^2}\, dx$

$= \frac{3}{2} \int g' e^{g}\, dx$

$= \frac{3}{2} e^{g} + C$

$= \frac{3}{2} e^{x^2} + C$

(m) $\int 2^x(2^x+3)^3\,dx$ $\qquad g(x) = 2^x + 3 \qquad g'(x) = 2^x \ln 2$

$= \frac{1}{\ln 2} \int 2^x \ln 2\, (2^x+3)^3\, dx$

$= \frac{1}{\ln 2} \int g' g^3\, dx$

$= \frac{1}{\ln 2} \frac{g^4}{4} + C$

$= \frac{1}{4\ln 2} \cdot (2^x+3)^4 + C$

(n) $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$

$= \displaystyle\int \frac{x}{\sqrt{1-(x^2)^2}}\,dx$ $\qquad g(x) = x^2 \qquad g'(x) = 2x$

$= \displaystyle\frac{1}{2}\int \frac{2x}{\sqrt{1-(x^2)^2}}\,dx$

$= \displaystyle\frac{1}{2}\int \frac{g'}{\sqrt{1-g^2}}\,dx$

$= \displaystyle\frac{1}{2}\arcsin g + C$

$= \displaystyle\frac{1}{2}\arcsin x^2 + C$

(o) $\displaystyle\int \frac{\ln x}{x}\,dx$ $\qquad g(x) = \ln x \qquad g'(x) = \frac{1}{x}$

$= \displaystyle\int g \cdot g'\,dx$

$= \displaystyle\frac{g^2}{2} + C$

$= \displaystyle\frac{1}{2}\ln^2|x| + C$

(p) $\int \sin^2 x \cos x \, dx$ $\quad g(x) = \sin x \quad g'(x) = \cos x$

$$= \int g^2 g' \, dx$$

$$= \frac{g^3}{3} + C$$

$$= \frac{\sin^3 x}{3} + C$$

(q) $\int \frac{\sin 3x}{\sqrt[3]{\cos^4 3x}} \, dx$ $\quad g(x) = \cos 3x \quad g'(x) = -3 \sin 3x$

$$= -\frac{1}{3} \int \frac{-3 \sin 3x}{\sqrt[3]{\cos^4 3x}} \, dx$$

$$= -\frac{1}{3} \int g' \, g^{-\frac{4}{3}} \, dx$$

$$= -\frac{1}{3} \cdot \frac{g^{-\frac{1}{3}}}{-\frac{1}{3}} + C$$

$$= \frac{1}{\sqrt[3]{g}} + C$$

$$= \frac{1}{\sqrt[3]{\cos 3x}} + C$$

3. Calculer les primitives suivantes :

(a) $\displaystyle\int x\sqrt{1-x}\,dx$ On pose $t = 1-x \Leftrightarrow x = 1-t$

$\Leftrightarrow dx = -dt$

$$= -\int (1-t)\sqrt{t}\,dt$$

$$= -\int \sqrt{t}\,dt + \int t\sqrt{t}\,dt$$

$$= -\int t^{\frac{1}{2}}\,dt + \int t^{\frac{3}{2}}\,dt$$

$$= -\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{5}{2}}}{\frac{5}{2}} + C$$

$$= -\frac{2}{3}\sqrt{t^3} + \frac{2}{5}\sqrt{t^5} + C$$

$$= -\frac{2}{3}\sqrt{(1-x)^3} + \frac{2}{5}\sqrt{(1-x)^5} + C$$

(b) $\displaystyle\int x^2(x-3)^5\,dx$ On pose $t = x-3 \Leftrightarrow x = t+3$

$\Leftrightarrow dx = dt$

$$= \int (t+3)^2 t^5\,dt = \int (t^2+6t+9)\,t^5\,dt$$

$$= \int t^7\,dt + 6\int t^6\,dt + 9\int t^5\,dt$$

$$= \frac{t^8}{8} + 6\,\frac{t^7}{7} + 9\,\frac{t^6}{6} + C$$

$$= \frac{(x-3)^8}{8} + 6\,\frac{(x-3)^7}{7} + 3\,\frac{(x-3)^6}{2} + C$$

(c) $\displaystyle\int \frac{3x}{(5x+3)^2}\,dx$

On pose $t = 5x + 3 \Leftrightarrow x = \frac{t-3}{5}$

$\Leftrightarrow dx = \frac{1}{5}\,dt$

$$= \int \frac{3\,\frac{t-3}{5}}{t^2}\,\frac{1}{5}\,dt = \frac{3}{25}\int \frac{t-3}{t^2}\,dt$$

$$= \frac{3}{25}\int \frac{1}{t}\,dt - \frac{9}{25}\int \frac{1}{t^2}\,dt$$

$$= \frac{3}{25}\ln|t| - \frac{9}{25}\,\frac{t^{-1}}{-1} + C$$

$$= \frac{3}{25}\ln|5x+3| + \frac{9}{25}\cdot\frac{1}{5x+3} + C$$

(d) $\displaystyle\int (1+x)\sqrt{2x+3}\,dx$

On pose $t = 2x + 3 \Leftrightarrow x = \frac{t-3}{2}$

$\Leftrightarrow dx = \frac{1}{2}\,dt$

$$= \int \left(1 + \frac{t-3}{2}\right)\sqrt{t}\cdot\frac{1}{2}\,dt$$

$$= \frac{1}{2}\int \frac{t-1}{2}\sqrt{t}\,dt = \frac{1}{4}\int t\sqrt{t} - \frac{1}{4}\int \sqrt{t}\,dt$$

$$= \frac{1}{4}\,\frac{2}{5}\sqrt{t^5} - \frac{1}{4}\cdot\frac{2}{3}\sqrt{t^3} + C$$

$$= \frac{1}{10}\sqrt{(2x+3)^5} - \frac{1}{6}\sqrt{(2x+3)^3} + C$$

(e) $\displaystyle\int \frac{x}{\sqrt[4]{3+4x}}\,dx$

On pose $t = 3+4x \iff x = \frac{t-3}{4}$

$\iff dx = \frac{1}{4}\,dt$

$$= \frac{1}{16}\int \frac{t-3}{\sqrt[4]{t}}\,dt = \frac{1}{16}\int t^{\frac{3}{4}}\,dt - \frac{3}{16}\int t^{-\frac{1}{4}}\,dt$$

$$= \frac{1}{16}\cdot\frac{4}{7}\sqrt[4]{t^7} - \frac{3}{16}\,\frac{4}{3}\sqrt[4]{t^3} + C$$

$$= \frac{1}{28}\sqrt[4]{(3+4x)^7} - \frac{1}{4}\sqrt[4]{(3+4x)^3} + C$$

(f) $\displaystyle\int x(2x+1)^6\,dx$

On pose $t = 2x+1 \iff x = \frac{t-1}{2}$

$\iff dx = \frac{1}{2}\,dt$

$$= \frac{1}{4}\int (t-1)\,t^6\,dt$$

$$= \frac{1}{4}\int t^7\,dt - \frac{1}{4}\int t^6\,dt$$

$$= \frac{1}{4}\,\frac{t^8}{8} - \frac{1}{4}\,\frac{t^7}{7} + C$$

$$= \frac{1}{32}(2x+1)^8 - \frac{1}{28}(2x+1)^7 + C$$

~~3~~ 4. Calculer les primitives suivantes :

(a) $\int x^2 e^x \, dx$

$f = x^2 \qquad f' = 2x$

$g' = e^x \qquad g = e^x$

$\int x^2 e^x \, dx = x^2 e^x - 2\int x e^x \, dx$

$f = x \qquad f' = 1$

$g' = e^x \qquad g = e^x$

$I = x^2 e^x - 2\left[x e^x - \int e^x \, dx\right]$

$= x^2 e^x - 2x e^x + 2e^x + C$

$= e^x (x^2 - 2x + 2) + C$

(b) $\int (x+1)\ln x \, dx$

$f = \ln x \qquad f' = \frac{1}{x}$

$g' = x+1 \qquad g = \frac{x^2}{2} + x$

$$I = \left(\frac{x^2}{2} + x\right)\ln x - \int \frac{1}{x}\left(\frac{x^2}{2} + x\right) dx$$

$$= \left(\frac{x^2}{2} + x\right)\ln x - \int \left(\frac{x}{2} + 1\right) dx$$

$$= \left(\frac{x^2}{2} + x\right)\ln x - \frac{x^2}{4} - x + C$$

(c) $\int \ln x \, dx$

$f = \ln x \qquad f' = \frac{1}{x}$

$g' = 1 \qquad g = x$

$$\int \ln x \, dx = x \ln x - \int dx$$

$$= x \ln x - x + c$$

(d) $\int (x+2)e^{-x}\,dx$

$$\begin{cases} f = x+2 & f' = 1 \\ g' = e^{-x} & g = -e^{-x} \end{cases}$$

$$\int (x+2)e^{-x}\,dx = -(x+2)e^{-x} + \int e^{-x}\,dx$$

$$= -(x+2)e^{-x} - e^{-x} + C$$

$$= -(x+2+1)e^{-x} + C$$

$$= -(x+3)e^{-x} + C$$

(e) $\int \frac{\arcsin \sqrt{x}}{\sqrt{x}}\, dx$

$f = \arcsin \sqrt{x}$ $\qquad$ $f' = \frac{1}{2\sqrt{x}\sqrt{1-x}}$

$g' = \frac{1}{\sqrt{x}}$ $\qquad$ $g = 2\sqrt{x}$

$$\int \frac{\arcsin \sqrt{x}}{\sqrt{x}}\, dx = 2\sqrt{x} \arcsin \sqrt{x} \ \ldots$$

$$+ \int \frac{-1}{\sqrt{1-x}}\, dx$$

$$= 2\sqrt{x} \arcsin \sqrt{x} + 2\sqrt{1-x} + C$$

(f) $\int \cos x . e^x \, dx$

$f = \cos x \qquad f' = -\sin x$

$g' = e^x \qquad g = e^x$

$$\int \cos x \, e^x \, dx = \cos x \, e^x + \int \sin x \, e^x \, dx$$

$f = \sin x \qquad f' = \cos x$

$g' = e^x \qquad g = e^x$

$$= \cos x \, e^x + \sin x \, e^x - \int e^x \cos x \, dx$$

$$\Leftrightarrow \quad 2 \int e^x \cos x \, dx = (\cos x + \sin x) e^x + C$$

$$\Leftrightarrow \quad \int e^x \cos x \, dx = \frac{1}{2} (\sin x + \cos x) \, e^x + C$$

5~~1~~. Calculer les primitives suivantes :

(a) $\displaystyle\int \frac{1}{x^2-6x+8}\,dx$

$D \stackrel{\Delta}{=} (x-2)(x-4)$

$$\frac{1}{x^2-6x+8} = \frac{A}{x-2} + \frac{B}{x-4} \iff 1 = A(x-4)+B(x-2)$$

- $x = 4 \iff 1 = 2B$

  $\iff B = \frac{1}{2}$

- $x = 2 \iff 1 = -2A$

  $\iff A = -\frac{1}{2}$

$$\Rightarrow \int \frac{1}{x^2-6x+8}\,dx = -\frac{1}{2}\int \frac{1}{x-2}\,dx + \frac{1}{2}\int \frac{1}{x-4}\,dx$$

$$= -\frac{1}{2}\ln|x-2| + \frac{1}{2}\ln|x-4| + c$$

$$= \frac{1}{2}\ln\left|\frac{x-4}{x-2}\right| + C$$

(b) $\displaystyle\int \frac{1}{2x^2-5x-3}\,dx$ $\qquad D \stackrel{\Delta}{=} (2x+1)(x-3)$

$$\frac{1}{2x^2-5x-3} = \frac{A}{2x+1} + \frac{B}{x-3}$$

$$\Rightarrow \quad 1 = A(x-3) + B(2x+1)$$

- $x = 3 \iff 1 = 7B \iff B = \frac{1}{7}$
- $x = -\frac{1}{2} \iff 1 = -\frac{7A}{2} \iff A = -\frac{2}{7}$

$$\int \frac{1}{2x^2-5x-3}\,dx = -\frac{2}{7}\frac{1}{2}\int \frac{2\,dx}{2x+1} + \frac{1}{7}\int \frac{dx}{x-3}$$

$$= -\frac{1}{7}\ln|2x+1| + \frac{1}{7}\ln|x-3| + C$$

$$= \frac{1}{7}\ln\left|\frac{x-3}{2x+1}\right| + C$$

(c) $\displaystyle\int \frac{1}{x^3-5x^2+2x+8}\,dx$ $\qquad D \overset{\triangle}{\underset{(M)}{=}} (x+1)(x-2)(x-4)$

$$\frac{1}{D} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x-4}$$

$$\Leftrightarrow 1 = A(x-2)(x-4) + B(x+1)(x-4) + C(x+1)(x-2)$$

- $x = 2 \Leftrightarrow 1 = -6B \Leftrightarrow B = -\frac{1}{6}$
- $x = 4 \Leftrightarrow 1 = 10C \Leftrightarrow C = \frac{1}{10}$
- $x = -1 \Leftrightarrow 1 = 15A \Leftrightarrow A = \frac{1}{15}$

$$\int \frac{1}{\cdots}\,dx = \frac{1}{15}\int \frac{1}{x+1}\,dx - \frac{1}{6}\int \frac{dx}{x-2} + \frac{1}{10}\int \frac{dx}{x-4}$$

$$= \frac{1}{15}\ln|x+1| - \frac{1}{6}\ln|x-2| + \ldots$$

$$\ldots \frac{1}{10}\ln|x-4| + C$$

$$\left(= \frac{2\ln|x+1| - 5\ln|x-2| + 3\ln|x-4|}{30} + C \right.$$

$$\left. = \frac{1}{30}\ln\frac{(x+1)^2|x-4|^3}{|x-2|^5} + C \right)$$

(d) $\displaystyle\int \frac{8}{x^3-2x^2+4x-8}\,dx$ $\qquad D \overset{\text{(H)}}{=} (x-2)(x^2+4)$

$$\frac{8}{D} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$$

$$\Leftrightarrow \quad 8 = A(x^2+4) + (Bx+C)(x-2)$$

- $x = 2 \Leftrightarrow 8 = 8A \Leftrightarrow A = 1$
- $x = 0 \Leftrightarrow 8 = 4 - 2C \Leftrightarrow C = -2$
- $x = 1 \Leftrightarrow 8 = 5 + (B-2)(-1)$

$$\Leftrightarrow 1 = -B \Leftrightarrow B = -1$$

$$\int \frac{8}{\cdots}\,dx = \int \frac{dx}{x-2} + \int \frac{-x-2}{x^2+4}\,dx$$

$$= \ln|x-2| - \frac{1}{2}\int \frac{2x}{x^2+4}\,dx - 2\underbrace{\int \frac{dx}{x^2+4}}$$

$$= \ln|x-2| - \frac{1}{2}\ln(x^2+4) - 2I_3 \quad I_3$$

$$I_3 = \int \frac{dx}{x^2+4} = \int \frac{dx}{4\left(\frac{x^2}{4}+1\right)} = \frac{1}{4}\,2\int \frac{\frac{1}{2}\,dx}{\left(\frac{x}{2}\right)^2+1}$$

$$= \frac{1}{2}\arctan\frac{x}{2} + C$$

$$\Rightarrow \int \frac{8}{\cdots}\,dx = \ln|x-2| - \frac{1}{2}\ln(x^2+4) - \arctan\frac{x}{2} + C$$

(e) $\displaystyle\int \frac{1}{x^2+2x+5}\,dx$

$\Delta_D < 0 \Rightarrow D$ non fact.

$$= \int \frac{1}{(x^2+2x+1)+4}\,dx$$

$$\Rightarrow \int \frac{g'}{g^2+1}\,dx = \arctan g + C$$

$$= \int \frac{1}{(x+1)^2+4}\,dx$$

$$= \int \frac{dx}{4\left[\frac{(x+1)^2}{4}+1\right]}$$

$$= \frac{1}{4}\, 2 \int \frac{\overbrace{\tfrac{1}{2}}^{g'(x)}\,dx}{\Big(\underbrace{\tfrac{x+1}{2}}_{g(x)}\Big)^2+1}$$

$$= \frac{1}{2} \arctan g(x) + C$$

$$= \frac{1}{2} \arctan \frac{x+1}{2} + C$$

(f) $\displaystyle\int \frac{x}{x^2+4x+6}\,dx$ $\Delta_D < 0 \Rightarrow D$ non fact.

$$= \frac{1}{2}\int \frac{2x+4-4}{x^2+4x+6}\,dx$$ Faire apparaître la dérivée de D ou N

$$= \frac{1}{2}\int \frac{\overbrace{2x+4}^{g'(x)}}{\underbrace{x^2+4x+6}_{g(x)}}\,dx - \frac{1}{2}\int \frac{4}{x^2+4x+6}\,dx$$

↳ arctan g (cf (e))

$$= \frac{1}{2}\ln(x^2+4x+6) - 2\underbrace{\int \frac{1}{(x^2+4x+4)+2}\,dx}_{I_2}$$

$$I_2 = \int \frac{1}{(x+2)^2+2}\,dx = \int \frac{1}{2\left[\frac{(x+2)^2}{2}+1\right]}\,dx$$

$$= \frac{1}{2}\sqrt{2}\int \frac{\overbrace{\frac{1}{\sqrt{2}}}^{g'}\,1}{\Big(\underbrace{\frac{x+2}{\sqrt{2}}}_{g}\Big)^2+1}\,dx = \frac{\sqrt{2}}{2}\arctan\left(\frac{x+2}{\sqrt{2}}\right)+C$$

$$\Rightarrow I = \frac{1}{2}\ln(x^2+4x+6) - \sqrt{2}\arctan\frac{\sqrt{2}(x+2)}{2} + C$$

(g) $\displaystyle\int \frac{6x^4 - 5x^3 + 4x^2}{2x^2 - x + 1}\,dx$

• $d^\circ N > d^\circ D \longrightarrow$ division euclidienne

$$\begin{array}{rl|l} & 6x^4 - 5x^3 + 4x^2 & 2x^2 - x + 1 \\ - & (6x^4 - 3x^3 + 3x^2) & 3x^2 - x \\ \hline & -2x^3 + x^2 & \\ - & (-2x^3 + x^2 - x) & \\ \hline & x & \end{array}$$

• $I = \displaystyle\int (3x^2 - x)\,dx + \int \frac{x}{2x^2 - x + 1}\,dx$

$= 3\dfrac{x^3}{3} - \dfrac{x^2}{2} + \dfrac{1}{4}\displaystyle\int \frac{4x - 1 + 1}{2x^2 - x + 1}\,dx$

$= x^3 - \dfrac{x^2}{2} + \dfrac{1}{4}\displaystyle\int \frac{4x - 1}{2x^2 - x + 1}\,dx \dots$

$\dots + \dfrac{1}{4}\underbrace{\displaystyle\int \frac{dx}{2x^2 - x + 1}}_{I_1}$

$= x^3 - \dfrac{x^2}{2} + \dfrac{1}{4}\ln(2x^2 - x + 1) \dots$

$\dots + \dfrac{1}{4} I_1$

$$\bullet\ I_1 = \frac{1}{2}\int \frac{dx}{x^2 - \frac{x}{2} + \left(\frac{1}{4}\right)^2 + \frac{1}{2} - \left(\frac{1}{4}\right)^2}$$

$$= \frac{1}{2}\int \frac{dx}{\left(x - \frac{1}{4}\right)^2 + \frac{7}{16}}$$

$$= \frac{1}{2}\int \frac{dx}{\frac{7}{16}\left[\frac{16}{7}\left(x - \frac{1}{4}\right)^2 + 1\right]}$$

$$= \frac{1}{2} \cdot \frac{16}{7}\int \frac{dx}{\left[\frac{4}{\sqrt{7}}\left(x - \frac{1}{4}\right)\right]^2 + 1}$$

$$= \frac{8}{7}\frac{\sqrt{7}}{4}\int \frac{\frac{4}{\sqrt{7}}\,dx}{\left(\frac{4x-1}{\sqrt{7}}\right)^2 + 1}$$

$$= \frac{2\sqrt{7}}{7}\arctan\left(\frac{4x-1}{\sqrt{7}}\right) + C$$

$$\bullet\ I = x^3 - \frac{x^2}{2} + \frac{1}{4}\ln\left(2x^2 - x + 1\right) \ldots$$

$$+ \frac{\sqrt{7}}{14}\arctan\frac{\sqrt{7}\,(4x-1)}{7} + c$$

5. Calculer les primitives suivantes :

(a) $\int (e^{3x} - x)^2 \, dx$

$$= \int (e^{6x} - 2xe^{3x} + x^2) \, dx$$

$$= \frac{1}{6} e^{6x} - 2 \int x e^{3x} \, dx + \frac{x^3}{3}$$

$f = x \qquad f' = 1$

$g' = e^{3x} \qquad g = \frac{1}{3} e^{3x}$

$$= \frac{1}{6} e^{6x} - \frac{2}{3} x e^{3x} + \frac{1}{3} \int e^{3x} \, dx + \frac{x^3}{3}$$

$$= \frac{1}{6} e^{6x} - \frac{2}{3} x e^{3x} + \frac{1}{9} e^{3x} + \frac{x^3}{3} + C$$

(b) $\int \frac{e^x}{\sqrt{1 - e^{2x}}} \, dx$

$g = e^x \qquad g' = e^x$

$$= \int \frac{g'}{\sqrt{1 - g^2}} \, dx = \arcsin g + C$$

$$= \arcsin e^x + C$$

(c) $\int \frac{e^x}{e^{2x}+2e^x+2}\,dx = \int \frac{e^x}{(e^{2x}+2e^x+1)+1}\,dx$

$= \int \frac{e^x}{(e^x+1)^2+1}\,dx$

$= \arctan(e^x+1) + C$

(d) $\int \frac{\sin x}{1-\sin^2 x}\,dx = \int \frac{\sin x}{\cos^2 x}\,dx$

$= -\int -\sin x \cos^{-2} x \,dx = -\frac{\cos^{-1} x}{-1} + C$

$= +\frac{1}{\cos x} + C$

(e) $\displaystyle\int \frac{\ln x}{x^5}\,dx$

$f = \ln x \longrightarrow f' = \frac{1}{x}$

$g' = \frac{1}{x^5} = x^{-5} \longrightarrow g = \frac{-1}{4x^4}$ (*)

(*)

$$= -\frac{\ln x}{4x^4} + \frac{1}{4}\int \frac{1}{x^5}\,dx$$

$$= -\frac{\ln x}{4x^4} + \frac{1}{16}\,\frac{1}{x^4} + c$$

(f) $\displaystyle\int \sin 2x\, e^{3x-2}\,dx$

$f = \sin 2x \longrightarrow f' = 2\cos 2x$

$g' = e^{3x-2} \longrightarrow g = \frac{1}{3}e^{3x-2}$

$$= \frac{1}{3}e^{3x-2}\sin 2x - \frac{2}{3}\int \cos 2x\, e^{3x-2}\,dx$$

$f = \cos 2x \longrightarrow f' = -2\sin 2x$

$g' = e^{3x-2} \longrightarrow g = \frac{1}{3}e^{3x-2}$

$$I = \frac{1}{3}e^{3x-2}\sin 2x - \frac{2}{3}\left[\frac{1}{3}\cos 2x\, e^{3x-2} + \frac{2}{3}\underbrace{\int \sin 2x\, e^{2x-3}\,dx}_{I}\right]$$

$$= \frac{1}{3}e^{3x-2}\sin 2x - \frac{2}{9}\cos 2x\, e^{3x-2} - \frac{4}{9}I$$

$$\Leftrightarrow I + \frac{4}{9}I = \frac{e^{2x-3}}{3}\left(\sin 2x - \frac{2}{3}\cos 2x\right) + C$$

$$\Leftrightarrow \frac{9}{13}\,\frac{13}{9}\,I = (\ldots)\cdot\frac{9}{13}$$

$$\Leftrightarrow I = \frac{\cancel{9}}{13}\cdot\frac{e^{2x-3}}{\cancel{3}}\;\frac{3\sin 2x - 2\cos 2x}{\cancel{3}} + C$$

$$= \frac{1}{13}\,e^{2x-3}\left(3\sin 2x - 2\cos 2x\right) + C$$

(g) $\displaystyle\int x \ln x^3 \, dx$

$f = \ln x^3 \rightarrow f' = \frac{3x^2}{x^3} = \frac{3}{x}$

$g' = x \rightarrow g = \frac{x^2}{2}$

$$= \frac{x^2 \ln x^3}{2} - \frac{3}{2}\int x \, dx$$

$$= \frac{x^2 \ln x^3}{2} - \frac{3}{4} x^2 + C$$

(h) $\displaystyle\int \frac{e^{2x}}{\sqrt{e^x+1}} \, dx$

On pose $t = e^x + 1 \rightarrow e^x = t - 1$

$dt = e^x \, dx$

$$= \int \frac{e^x \cdot e^x \, dx}{\sqrt{e^x+1}} = \int \frac{t-1}{\sqrt{t}} \, dt$$

$$= \int \frac{t}{\sqrt{t}} \, dt - \int \frac{1}{\sqrt{t}} \, dt$$

$$= \int \sqrt{t} \, dt - \int t^{-\frac{1}{2}} \, dt$$

$$= \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \frac{2\sqrt{t^3}}{3} - 2\sqrt{t} + C$$

$$= \frac{2}{3}\sqrt{(e^x+1)^3} - 2\sqrt{e^x+1} + C$$

(i) $\displaystyle\int \frac{1}{x^2+2x}\,dx$ $\qquad D = x(x+2)$

$$\frac{1}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2} \iff 1 = A(x+2) + B(x)$$

$x = -2 \to B = -\frac{1}{2}$

$x = 0 \to A = \frac{1}{2}$

$$\int \frac{1}{x(x+2)} = \frac{1}{2}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x+2}$$

$$= \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2| + C$$

$$= \frac{1}{2}\ln\left|\frac{x}{x+2}\right| + C$$

(j) $\displaystyle\int \frac{1}{3x^2-x+1}\,dx$ $\qquad \Delta_D < 0 \to D$ non fact. $\to$ arctan g

$$= \int \frac{1}{3\left(x^2 - \frac{x}{3} + \frac{1}{3}\right)}\,dx = \frac{1}{3}\int \frac{dx}{\left(x^2 - \frac{x}{3} + \frac{1}{36}\right) + \frac{1}{3} - \frac{1}{36}}$$

$$= \frac{1}{3}\int \frac{dx}{\left(x - \frac{1}{6}\right)^2 + \frac{11}{36}} = \frac{1}{3}\int \frac{dx}{\frac{11}{36}\left[\frac{36}{11}\left(x - \frac{1}{6}\right)^2 + 1\right]}$$

$$= \frac{1}{3}\cdot\frac{36}{11}\int \frac{dx}{\left[\frac{\cancel{6}}{\sqrt{11}}\left(\frac{6x-1}{\cancel{6}}\right)\right]^2 + 1} = \frac{12}{11}\frac{\sqrt{11}}{6}\int \frac{\frac{6}{\sqrt{11}}\,dx}{\left(\frac{6x-1}{\sqrt{11}}\right)^2 + 2}$$

$$= \frac{2\sqrt{11}}{11}\arctan\left[\sqrt{11}\,\frac{6x-1}{11}\right] + C$$

(k) $\displaystyle\int \left(a + \frac{b}{x-a}\right)^2 dx$

$$= \int \left(a^2 + \frac{2ba}{x-a} + \frac{b^2}{(x-a)^2} \right) dx$$

$$= a^2 x + 2ab \ln|x-a| + b^2 \int (x-a)^{-2} dx$$

$$= a^2 x + 2ab \ln|x-a| - b^2 \frac{1}{x-a} + C$$

(l) $\displaystyle\int \overbrace{\frac{1}{x \underbrace{\ln^2 x}_{g = \ln x}}}^{g'} dx$

$$= \int \frac{g'}{g^2} dx = \int g' g^{-2} dx = \frac{g^{-1}}{-1} + C$$

$$= -\frac{1}{g} + C = -\frac{1}{\ln x} + C$$

(m) $\displaystyle\int \frac{\overbrace{1 - \sin x}^{g'}}{\underbrace{x + \cos x}_{g}} dx$

$$= \int \frac{g'}{g} dx = \ln|g| + C$$

$$= \ln|x + \cos x| + C$$

(n) $\displaystyle\int \frac{5-3x}{\sqrt{4-3x^2}}\,dx$

$$= \int \frac{5}{\sqrt{4-3x^2}}\,dx - 3\int \frac{x}{\sqrt{4-3x^2}}\,dx$$

$$= I_1 - I_2$$

$$I_1 = 5\int \frac{1}{2\sqrt{1-\frac{3x^2}{4}}}\,dx = \frac{5}{2}\int \frac{dx}{\sqrt{1-\left(\frac{\sqrt{3}x}{2}\right)^2}}$$

$$= \frac{5}{2}\,\frac{2}{\sqrt{3}}\int \frac{\frac{\sqrt{3}}{2}\,dx}{\sqrt{1-\left(\frac{\sqrt{3}x}{2}\right)^2}} = \frac{5}{\sqrt{3}}\arcsin\frac{\sqrt{3}x}{2} + C$$

$$I_2 = 3\,\frac{1}{-6}\int \frac{\overbrace{-6x}^{g'}\,dx}{\sqrt{\underbrace{4-3x^2}_{g}}} = -\frac{1}{2}\int g' g^{-\frac{1}{2}}\,dx$$

$$= -\frac{1}{2}\,\frac{\sqrt{g}}{\frac{1}{2}} + C = -\sqrt{4-3x^2}$$

$$\Rightarrow I = \frac{5\sqrt{3}}{3}\arcsin\frac{\sqrt{3}x}{2} + \sqrt{4-3x^2} + C$$

(o) $\displaystyle\int \frac{(1+x)^2}{x(1+x^2)}\,dx$

$$\frac{(1+x)^2}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}$$

$$\Leftrightarrow (1+x)^2 = A(1+x^2) + (Bx+C)x$$

$$x=0 \Leftrightarrow 1 = A$$

$$x=1 \Leftrightarrow 4 = 2A + B + C \Leftrightarrow B+C = 2 \quad (1)$$

$$x=-1 \Leftrightarrow 0 = 2A + B - C \Leftrightarrow B-C = -2 \quad (2)$$

$$(1)+(2) \Leftrightarrow 2B = 0 \Leftrightarrow B = 0 \Leftrightarrow C = 2$$

$$I = \int \frac{dx}{x} + \int \frac{2dx}{1+x^2} = \ln|x| + 2\arctan x + C$$

(p) $\displaystyle\int \frac{\arcsin^2 x}{\sqrt{1-x^2}}\,dx$ $\qquad g = \arcsin x \qquad g' = \frac{1}{\sqrt{1-x^2}}$

$$I = \int g^2 g' \, dx = \frac{g^3}{3} + C = \frac{\arcsin^3 x}{3} + C$$

(q) $\displaystyle\int \frac{\sin^3 x}{\sqrt{\cos x}}\,dx = \int \frac{\sin x \sin^2 x}{\sqrt{\cos x}}\,dx$

$$= \int \frac{\sin x\,(1-\cos^2 x)}{\sqrt{\cos x}}\,dx$$

$$= \int \frac{\overset{-g'}{\sin x}}{\sqrt{\underset{g}{\cos x}}}\,dx - \int \underbrace{\frac{\cos^2 x}{\sqrt{\cos x}}}_{g^{3/2}} \underset{-g'}{\sin x}\,dx$$

$$= -\int g^{-\frac{1}{2}} g' \, dx + \int g^{\frac{3}{2}} g' \, dx$$

$$= -\frac{g^{\frac{1}{2}}}{\frac{1}{2}} + \frac{g^{\frac{5}{2}}}{\frac{5}{2}} + C$$

$$= -2\sqrt{\cos x} + 2\,\frac{\sqrt{\cos^5 x}}{5} + C$$

(r) $\displaystyle\int \frac{1}{\sqrt{2-3x-x^2}}\,dx$ $\longrightarrow$ arcsin g

$$= \int \frac{1}{\sqrt{2+\frac{9}{4} - \left(x^2+3x+\frac{9}{4}\right)}}\,dx$$

$$= \int \frac{1}{\sqrt{\frac{17}{4} - \left(x+\frac{3}{2}\right)^2}}\,dx$$

$$= \int \frac{1}{\sqrt{\frac{17}{4}}\ \sqrt{1-\frac{4}{17}\left(x+\frac{3}{2}\right)^2}}\,dx$$

$$= \frac{2}{\sqrt{17}} \int \frac{1}{\sqrt{1-\left[\frac{\cancel{2}}{\sqrt{17}}\left(\frac{2x+3}{\cancel{2}}\right)\right]^2}}\,dx$$

$$= \frac{2\sqrt{17}}{17}\,\frac{\sqrt{17}}{2} \int \frac{\frac{2}{\sqrt{17}}\ \ 1}{\sqrt{1-\left(\frac{2x+3}{\sqrt{17}}\right)^2}}\,dx$$

$$= \arcsin \frac{\sqrt{17}\,(2x+3)}{17} + C$$

(s) $\displaystyle\int (t^2+1)\ln(t^2+1)\,dt$

$$f = \ln(t^2+1) \qquad f' = \frac{2t}{t^2+1}$$

$$g' = t^2+1 \qquad g = \frac{t^3}{3} + t$$

$$I = \left(\frac{t^3}{3}+t\right)\ln(t^2+1) - 2\underbrace{\int \frac{t}{t^2+1}\left(\frac{t^3}{3}+t\right)dt}_{I_1}$$

$$I_1 = \int \frac{\frac{t^4}{3}+t^2}{t^2+1}\,dt$$

div euclidienne $\rightarrow \dfrac{\frac{t^4}{3}+t^2}{t^2+1} = \dfrac{t^2}{3} + \dfrac{2}{3} - \dfrac{2}{3(t^2+1)}$

$$\rightarrow I_1 = \frac{t^3}{9} + \frac{2t}{3} - \frac{2}{3}\arctan t + C$$

$$\rightarrow I = \left(\frac{t^3}{3}+t\right)\ln(t^2+1) - \frac{2t^3}{9} - \frac{4t}{3} + \frac{4}{3}\arctan t + C$$

(t) $\displaystyle\int \frac{x \arcsin x}{\sqrt{1-x^2}}\,dx$

$f = \arcsin x \qquad f' = \frac{1}{\sqrt{1-x^2}}$

$g' = -\frac{1}{2}\,\frac{-2x}{\sqrt{1-x^2}} \qquad g = -\sqrt{1-x^2}$

$I = -\sqrt{1-x^2}\arcsin x + \int \frac{\cancel{\sqrt{1-x^2}}}{\cancel{\sqrt{1-x^2}}}\,dx$

$= x - \sqrt{1-x^2}\arcsin x + C$

(u) $\displaystyle\int x \arctan x\,dx$

$f = \arctan x \qquad f' = \frac{1}{1+x^2}$

$g' = x \qquad g = \frac{x^2}{2}$

$I = \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2}{\underbrace{1+x^2}}\,dx$

$\Leftrightarrow$ div. eucl. $= 1 - \frac{1}{1+x^2}$

$= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \left[1 - \frac{1}{1+x^2}\right] dx$

$= \frac{x^2}{2}\arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x + C$

$= \frac{x^2+1}{2}\arctan x - \frac{x}{2} + C$